Online Appendix

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Top-Cycles and Revealed Preference Structures

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Abstract

Following a topological approach, this appendix generalizes our main results—Theorems 5.2 and 5.3—to choice problems with infinitely many alternatives.

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In this appendix, we study rationalizability by preference structures for choice problems with infinitely many alternatives.

As usual, a *menu* refers to a nonempty set in a space X of alternatives. Consider a collection $\mathcal{D}$ of menus to act as the domain of our choice correspondence. A preorder $\succsim$ on X has the **maximal domination property** (MDP) **on** $\mathcal{D}$ if for each $S \in \mathcal{D}$ and $x \in S \backslash \mathbf{MAX}(S, \succsim)$, there exists a $z \in \mathbf{MAX}(S, \succsim)$ such that $z \succ x$. This property has a key role in the line of reasoning that connects some of our axioms to rationalization by preference structures. The conclusions of Propositions 4.1 and 4.2 also hinge upon MDP.

Clearly, any preorder satisfies MDP on the collection of all finite menus, $\mathfrak{X}$, but for larger collections MDP necessitates further assumptions on the structure of the preorder. At one extreme, we have Noetherian preorders that satisfy MDP on the collection of all menus. While our theory can be extended in this direction, we shall not do so here because such preorders often violate natural monotonicity properties (such as preferring more to less). Instead, we assume that X is a metric space, and focus on the collection of compact menus, denoted as $\mathfrak{X}_k$. This particular domain will help us relate MDP to topological properties of a preorder.

A binary relation $\mathbf{R}$ on (a metric space) X is said to be **open lower-semicontinuous** (OLSC) if $\{y \in X : x \ \mathbf{R}^{>} \ y\}$ is an open set for every $x \in X$. When $\mathbf{R}$ is complete, this property is equivalent to closedness of the weak upper section $\{y \in X : y \ \mathbf{R} \ x\}$ for each $x \in X$, which is a form of closed upper-semicontinuity. For an incomplete preorder, however, the latter is a distinct concept, and we prefer the former notion of continuity here because a choice correspondence is more closely associated with the strict part of a rationalizing core preference than its symmetric part (see Section 6 in the main text). Therefore, we will focus on **OLSC (weak) preference structures** $(\succsim, \mathbf{R})$ in which both $\mathbf{R}$ and $\succsim$ are OLSC binary relations. Such preference structures induce nonempty valued choice correspondences on $\mathfrak{X}_k$:

Lemma 1. *Let $(\succsim, \mathbf{R})$ be an OLSC weak preference structure on X. Then $\mathbf{MAX}(S, \succsim)$ is compact, and $\bigcirc(\mathbf{MAX}(S, \succsim), \mathbf{R})$ is nonempty for each $S \in \mathfrak{X}_k$.*

In a classical paper on representation of incomplete strict preferences, Peleg (1970) couples open continuity with a further topological property: A preorder $\succsim$ on X is **spacious** if $x \succ y$ and $z' \in \text{cl}\{z \in X : z \succ x\}$ jointly imply $z' \succ y$, where cl denotes topological closure.[1] In turn, we say that $\succsim$ is **weakly spacious** if for any convergent sequence (x_n) that is also $\succ$-increasing, we have $\lim x_n \succ x_1$. Here, "$\succ$-increasing" means $x_{n+1} \succ x_n$ for every $n \in \mathbb{N}$. Of course, a spacious preorder is also weakly spacious, but the converse does not hold in general.

The following lemma shows that MDP on $\mathfrak{X}_k$ and weak spaciousness are equivalent properties for an OLSC preorder.

Lemma 2. *An OLSC preorder $\succsim$ on a metric space X is weakly spacious if, and only if, it satisfies MDP on $\mathfrak{X}_k$.*

[1] More precisely, Peleg's definition of spaciousness focuses on the closure of the lower section $\{z \in X : x \succ z\}$, but the conclusion of his utility existence theorem remains valid with the dual version introduced above.

In what follows, a **choice correspondence on** $\mathfrak{X}_k$ refers to a function $C : \mathfrak{X}_k \to 2^X$ such that $\emptyset \neq C(S) \subseteq S$ for each $S \in \mathfrak{X}_k$. A weak preference structure $(\succsim, \mathbf{R})$ on X **rationalizes** such a choice correspondence if $C(S) = \bigcirc(\mathbf{MAX}(S, \succsim), \mathbf{R})$ for each $S \in \mathfrak{X}_k$. Some further definitions are naturally adopted to the domain $\mathfrak{X}_k$: A menu $S \in \mathfrak{X}_k$ is C-**irreducible** if for every $x, y \in S$, there is a menu $T \in \mathfrak{X}_k$ with $x \in T$ and $y \in C(T)$. If there is no such T for a pair of points $x, y \in X$, we say that x C-**dominates** y, also denoted as $x \succ_C y$. An asymmetric domination relation $\blacktriangleright_C$ is defined as $x \blacktriangleright_C y$ if and only if there exist $z \in X$ and $S \in \mathfrak{X}_k$ such that

$$z \in C(S \cup \{x\}), \quad y \notin C(S \cup \{x\}), \quad y \in C\{z, y\}, \quad \text{and} \quad y \in C(S). \tag{1}$$

Theorem 5.2 in the main text focuses on finite menus in order to guarantee MDP, which is indispensable for the validity of some of our axioms. Here, we will derive MDP from the topological properties of core preferences, as outlined above. Consequently, we can import all of the axioms from Theorem 5.2 to the present setting. For clarity, let us restate these axioms:

(H1) Preference Consistency. For every $S, T \in \mathfrak{X}_k$ such that $C\{x, y\} = \{x\}$ for every $(x, y) \in S \times T$, we have $C(S \cup T) \subseteq S$.

(H2) The Aizerman Condition. For every $S, T \in \mathfrak{X}_k$ with $S \subseteq T$ and $C(T) \subseteq S$, we have $C(S) = C(T)$.

(H3) The Binary Chernoff Axiom. For every $S, T \in \mathfrak{X}_k$ with $S \subseteq T$ and $|C(T)| \leq 2$, we have $C(T) \cap S \subseteq C(S)$.

(H4) Arrow's Undominated Choice Axiom. For every C-irreducible $S, T \in \mathfrak{X}_k$ with $S \subseteq T$ and $C(T) \cap S \neq \varnothing$, we have $C(T) \cap S \supseteq C(S)$.

As we have seen in Section 6, for a rationalizable choice correspondence C the relations $\blacktriangleright_C$ and $\succ_C$ are nested as follows:

$$x \blacktriangleright_C y \quad \Rightarrow \quad x \succ_C y. \tag{2}$$

It should be noted that the property (2) is closely related to (H4). To see this, pick any $x, y \in X$ such that $x \succ_C y$ does not hold. Then (2) entails that $x \blacktriangleright_C y$ is also false. This, in turn, means that for any $S \in \mathfrak{X}_k$ with $y \in C(S)$, if we let $T := S \cup \{x\}$, then y must also belong to $C(T)$ whenever the latter contains a point z such that $y \in C\{z, y\}$.

Beyond resemblance, it can be shown that property (2) is actually equivalent to Arrow's Undominated Choice Axiom in the statement of Theorem 5.2. That is, in the statement of Theorem 5.2, axiom (B2) can be replaced with property (2), without changing the conclusion of the theorem. This equivalence motivates a hybrid axiom that combines the behavioral content of (2) with a topological condition:

(H5) Dominance Consistency. For every $x, y \in X$ with $x \blacktriangleright_C y$, there exists an open set $U \subseteq X$ such that $y \in U$ and $x \succ_C U$.

To gain intuition, consider a pair of alternatives x and y with $x \blacktriangleright_C y$. By (1), this means that the decision maker selects y from a menu S, but the addition of x to the menu causes the decision maker to switch to some other alternative z that happens to be available in S, while also being revealed inferior to y in the binary menu $\{z, y\}$. So, the presence of x makes y less attractive for the decision maker without creating a similar effect on some other alternatives. We take this as evidence of a core preference: If $x \blacktriangleright_C y$, the core preference of the decision maker must rank x strictly above y. For this ranking to be compatible with open lower-semicontinuity, there must also exist a neighborhood U of y such that x is strictly better than all points in U, according to the core preference of the decision maker. Finally, as a minimum behavioral requirement for the latter notion, any point in U should never be selected when x is available, meaning $x \succ_C U$. This is the content of (H5).

We proceed with a novel axiom that addresses infinite menus:

(H6) Finite Confirmation. For every $S \in \mathfrak{X}_k$:
(i) If $z \in C(S)$ and $y \in S \backslash C(S)$, then there exists a finite menu $S' \subseteq S$ such that $z \in C(S')$ and $y \in S' \backslash C(S')$.
(ii) If $z, y \in C(S)$, then there exists a finite menu $S' \subseteq S$ such that $z, y \in C(S')$.

For a finite menu S, both parts of this axiom hold trivially. According to classical choice theory, if the decision maker strictly prefers z over y, this manifests itself with instances of the form $z \in C(S)$ and $y \in S \backslash C(S)$. Similarly, instances of the form $z, y \in C(S)$ are associated with the absence of strict preference between z and y. The Finite Confirmation Axiom states that if any of these two patterns is observed in an infinite menu, then the same pattern must also occur at least in one other instance with a finite subset of the feasible alternatives. Put differently, at least one finite subset must "confirm" the pattern observed in the original menu. Naturally, however, our model does not demand perfectly consistent behavior across all menus. For example, even if $z \in C(S)$ and $y \in S \backslash C(S)$ for a menu S, the decision maker may well select both z and y from another menu S'.

Recall that any compact set in X is the limit of a sequence of finite subsets (in Hausdorff distance). Thus, in classical models, a choice pattern observed in a given menu S is mimicked by finite subsets that are sufficiently close to S, at least if the pattern in question treats a given pair of alternatives asymmetrically. Part (i) of (H6) has a similar flavor, but it focuses on the collection of finite subsets in its entirety, as opposed to convergent sequences of subsets. Specifically, part (i) of (H6) can equivalently be stated as follows:

(H6) (i*) Given a pair of points $z, y \in X$, consider a menu S that contains both z and y. Suppose that $z \in C(S')$ implies $y \in C(S')$ for every finite menu $S' \subseteq S$ with $z, y \in S'$. Then $z \in C(S)$ implies $y \in C(S)$.

To understand why (H6) is useful for our purposes, note that the definition of the relation $\blacktriangleright_C$ is based on addition of a single alternative x to a given a menu S. Towards the characterization theorems that we seek, this necessitates inductive arguments based on successive addition/subtraction of alternatives. These inductive arguments work well only on finite menus, while (H6) allows us to focus on finite subsets of infinite menus.

We also need a sequential upper semi-continuity property for binary menus:

(H7) Binary Semicontinuity. Let (y_n) be a convergent sequence in X, and fix an $x \in X$. If $y_n \in C\{x, y_n\}$ for every $n \in \mathbb{N}$, then $\lim y_n \in C\{x, \lim y_n\}$.

We are now ready to state an analogue of Theorem 5.2.

Theorem 1. *Let X be a metric space. The following three statements are equivalent for a choice correspondence C on $\mathfrak{X}_k$.*
(a) *C satisfies Axioms* (H1)-(H3) *and* (H5)-(H7).
(b) *C is rationalizable by an open lower-semicontinuous preference structure $(\succsim, \mathbf{R})$ with a weakly spacious core preference $\succsim$.*
(c) *C is rationalizable by an open lower-semicontinuous weak preference structure $(\succsim, \mathbf{R})$ with a weakly spacious core preference $\succsim$.*

As its finite counterpart, the proof of rationalizability in Theorem 1 builds upon the notion of C-dominance, but we modify this relation topologically in order to obtain an OLSC core preference. Remarkably, this OLSC core preference also proves to be weakly spacious, as a consequence of the Aizerman Condition. This is the last step in the proof of the statement (a) $\Rightarrow$ (b) towards the end of this appendix, which illustrates how weak spaciousness/MDP is connected to our behavioral axioms.

Finally, an analogue of Theorem 5.3 follows from the classical version of the Chernoff Axiom, which also ensures (H1).

(H3') The Chernoff Axiom. For every $S, T \in \mathfrak{X}_k$ with $S \subseteq T$, we have $C(T) \cap S \subseteq C(S)$.

Theorem 2. *Let X be a metric space. The following three statements are equivalent for a choice correspondence C on $\mathfrak{X}_k$.*
(a) *C satisfies Axioms* (H2), (H3') *and* (H5)-(H7).
(b) *C is rationalizable by an open lower-semicontinuous and transitive weak preference structure $(\succsim, \mathbf{R})$ with a weakly spacious core preference $\succsim$.*
(c) *There exists an open lower-semicontinuous and weakly spacious preorder $\succsim$ on X such that $C(S) = \mathbf{MAX}(S, \succsim)$ for each $S \in \mathfrak{X}_k$.*

Proofs

Given a binary relation $\mathbf{Q}$ on X, for any $x \in X$ we set $x^{\uparrow}(\mathbf{Q}) := \{y \in X : y \,\mathbf{Q}\, x\}$ and $x_{\downarrow}(\mathbf{Q}) := \{y \in X : x \,\mathbf{Q}\, y\}$.

Proof of Lemma 1
Fix an $S \in \mathfrak{X}_k$. We shall first show that $\mathbf{MAX}(S, \succsim) \neq \varnothing$. Suppose by contradiction that for every $y \in S$, there exists an $x \in S$ with $x \succ y$. Then $S \subseteq \bigcup\{x_{\downarrow}(\succ) : x \in S\}$. Since $\succsim$ is OLSC, each $x_{\downarrow}(\succ)$ is an open set. From compactness of S, it follows that there exists a finite set $\{x^1, ..., x^n\} \subseteq S$ such that $S \subseteq \bigcup_{i=1}^n x^i_{\downarrow}(\succ)$. In particular, for any $j \in \{1, ..., n\}$, we have $x^j \in \bigcup_{i=1}^n x^i_{\downarrow}(\succ)$, which means $x^i \succ x^j$ for some $i \in \{1, ..., n\}$. But then the finite set $\{x^1, ..., x^n\}$ does not admit a $\succsim$-maximal point, which contradicts transitivity of $\succsim$. Hence, $\mathbf{MAX}(S, \succsim)$ is nonempty.

As we just noted, for any $y \in S \backslash \mathbf{MAX}(S, \succsim)$ and $x \in S$ with $x \succ y$, the set $U := x_{\downarrow}(\succ)$ is an open neighborhood of y. Moreover, $U \cap S$ is a subset of $S \backslash \mathbf{MAX}(S, \succsim)$ by definitions. So,

$S\backslash\mathbf{MAX}(S,\succsim)$ is a relatively open subset of S, which is equivalent to saying that $\mathbf{MAX}(S,\succsim)$ is relatively closed in S. Since S is compact, it follows that so is $\mathbf{MAX}(S,\succsim)$.

As $\mathbf{R}$ is open lower-semicontinuous and complete, the top-cycle $\bigcirc(S,\mathbf{R})$ is nonempty for each $S\in\mathfrak{X}_k$ (see, e.g., Duggan 2007). Then $\bigcirc(\mathbf{MAX}(S,\succsim),\mathbf{R})$ is also nonempty for each $S\in\mathfrak{X}_k$ because the set $\mathbf{MAX}(S,\succsim)$ belongs to $\mathfrak{X}_k$ as well.

Proof of Lemma 2

The following observation will help prove Lemma 2.

Lemma 3. *Let $\succsim$ be an OLSC preorder on a metric space X. Then for any $x\in X$ and $S\in\mathfrak{X}_k$, we have*

$$\mathbf{MAX}\left(\mathrm{cl}\{y\in S:y\succ x\},\succsim\right)\subseteq\mathbf{MAX}\left(S,\succsim\right).$$

Proof. Fix an $x\in X$. For brevity, let us write $\overline{x}^{\uparrow}$ in place of $\mathrm{cl}\{y\in S:y\succ x\}$. Since S is closed, it is clear that $\overline{x}^{\uparrow}\subseteq S$. Thus, a point $y'\in\overline{x}^{\uparrow}$ belongs to $\mathbf{MAX}(S,\succsim)$ iff there does not exist a $z\in S$ such that $z\succ y'$.

Now pick any $y'\in\mathbf{MAX}\big(\overline{x}^{\uparrow},\succsim\big)$, and suppose by contradiction that $z\succ y'$ for some $z\in S$. Since $\succsim$ is OLSC, this implies $z\succ U$ for an open set $U\subseteq X$ that contains y'. Moreover, y' belongs to $\overline{x}^{\uparrow}$, and the latter is the closure of $\{y\in S:y\succ x\}$. From the definition of the closure operator, it follows that the set $U\cap\{y\in S:y\succ x\}$ is nonempty. That is, there exists a $y\in U$ with $y\succ x$. But then $z\succ y\succ x$, which also implies $z\succ x$ by transitivity of $\succsim$. Thereby, we have found a point z in $\overline{x}^{\uparrow}$ that satisfies $z\succ y'$, which contradicts the hypothesis that $y'\in\mathbf{MAX}\big(\overline{x}^{\uparrow},\succsim\big)$. ■

We now turn to the proof of Lemma 2. Let $\succsim$ be an OLSC preorder on a metric space X. Suppose first that $\succsim$ satisfies MDP on $\mathfrak{X}_k$. Pick any convergent sequence (x_n) in X that is also $\succ$-increasing. Set $x:=\lim x_n$, and note that $S:=\{x\}\cup\{x_n:n\in\mathbb{N}\}$ is a compact set that belongs to $\mathfrak{X}_k$. Moreover, $\mathbf{MAX}(S,\succsim)=\{x\}$ because $x_{n+1}\succ x_n$ for each $n\in\mathbb{N}$. In particular, $x_1\in S\backslash\mathbf{MAX}(S,\succsim)$, and hence, MDP implies $z\succ x_1$ for some $z\in\mathbf{MAX}(S,\succsim)$. Since $\mathbf{MAX}(S,\succsim)=\{x\}=\{\lim x_n\}$, it follows that $\lim x_n\succ x_1$. So, $\succsim$ is weakly spacious as we sought.

To prove the converse, suppose now that $\succsim$ violates MDP on $\mathfrak{X}_k$. Then there exist $S\in\mathfrak{X}_k$ and $x\in S\backslash\mathbf{MAX}(S,\succsim)$ such that, for any $z\in X$,

$$z\succ x\quad\Rightarrow\quad z\notin\mathbf{MAX}(S,\succsim).\tag{3}$$

For $w\in X$, we shall write $\overline{w}^{\uparrow}$ in place of the set $\mathrm{cl}\{y\in S:y\succ w\}$. Let $x_1:=x$. Since x_1 belongs to $S\backslash\mathbf{MAX}(S,\succsim)$, the set $\overline{x}_1^{\uparrow}$ is nonempty. By Lemma 1, $\mathbf{MAX}(\overline{x}_1^{\uparrow},\succsim)$ is also nonempty and compact. Pick a point $y_2\in\mathbf{MAX}(\overline{x}_1^{\uparrow},\succsim)$. Let d denote a metric that generates the topology on X. Given that $y_2\in\overline{x}_1^{\uparrow}$, there exists a point $x_2\in S$ with $x_2\succ x_1$ and $d(x_2,y_2)<1/2$. Moreover, by (3), $x_2\succ x_1$ implies $x_2\notin\mathbf{MAX}(S,\succsim)$. So, x_2 belongs to $S\backslash\mathbf{MAX}(S,\succsim)$, and consequently the set $\overline{x}_2^{\uparrow}$ is also nonempty. A repetition of our previous arguments, then, yields a pair of points y_3, x_3 in S with $y_3\in\mathbf{MAX}(\overline{x}_2^{\uparrow},\succsim)$, $x_3\succ x_2$ and $d(x_3,y_3)<1/3$. Note also that $x_3\succ x_1$ because $\succsim$ is transitive, which implies $x_3\notin\mathbf{MAX}(S,\succsim)$ by (3). We can continue in this fashion to extract a pair of sequences (x_n) and (y_{n+1}) in S such that (x_n) is $\succ$-increasing, $y_{n+1}\in\mathbf{MAX}(\overline{x}_n^{\uparrow},\succsim)$ for every $n\in\mathbb{N}$, and $d(x_n,y_n)<1/n$ for $n\geq 2$.

Since S is compact, by passing to a subsequence if necessary, we can assume that (x_n) converges to a point x' in S. Then we also have $\lim y_n=x'$ because $\lim d(x_n,y_n)=0$ by construction. Moreover, by Lemma 3, the property $y_{n+1}\in\mathbf{MAX}(\overline{x}_n^{\uparrow},\succsim)$ implies $y_{n+1}\in\mathbf{MAX}(S,\succsim)$ for every n. As $\mathbf{MAX}(S,\succsim)$ is a closed subset of S, it follows that $x'\in\mathbf{MAX}(S,\succsim)$. But then, by (3), we cannot have $x' \succ x_1$. Thereby, we have constructed an $\succ$-increasing and convergent sequence (x_n) such that $\lim x_n \succ x_1$ is not true. So, the preorder $\succsim$ is not weakly spacious.

Proof of Theorem 1

(b) $\Rightarrow$ (a). Let $(\succsim, \mathbf{R})$ be an OLSC preference structure on X that rationalizes C on $\mathfrak{X}_k$. Assume further that $\succsim$ is weakly spacious. Then, by Lemma 2, $\succsim$ satisfies MDP on $\mathfrak{X}_k$. Consequently, that C satisfies the axioms (H1)-(H3) is proved by a straightforward modification of our corresponding arguments in the main text. The necessity of (H5) is also clear from Section 6 and the related discussion above.

To verify (H6), pick any $S \in \mathfrak{X}_k$ and $z, y \in S$. Assume first that $z \in C(S)$ and $y \notin C(S)$. If $y \in \mathbf{MAX}(S, \succsim)$, then $z\ \mathbf{R}^{>}\ y$ because $C(S)$ is an $\mathbf{R}$-highset in $\mathbf{MAX}(S, \succsim)$. Thus, in this case, with $S' := \{z, y\}$, we have $C(S') = \{z\}$. Since z and y are distinct, it also follows that $y \notin C(S')$. In turn, if $y \notin \mathbf{MAX}(S, \succsim)$, by MDP there exists a point $z^y \in \mathbf{MAX}(S, \succsim)$ such that $z^y \succ y$. Moreover, $z \in C(S)$ implies z tran$(\mathbf{R}|_{\mathbf{MAX}(S,\succsim)})$ z^y by Lemma 3.1 in the main text. That is, there exist finitely many points $z_0, z_1, ..., z_n$ in $\mathbf{MAX}(S, \succsim)$ such that $z = z_0\ \mathbf{R}\ z_1\ \mathbf{R} \cdots \mathbf{R}\ z_n = z^y$. Let $S' := \{z_0, ..., z_n\} \cup \{y\}$. By construction, we have $\mathbf{MAX}(S', \succsim) = \{z_0, ..., z_n\}$, and z_0 tran$(\mathbf{R}|_{\mathbf{MAX}(S',\succsim)})$ z_i for $1 \leq i \leq n$. Thus, $z = z_0$ belongs to $\bigcirc(\mathbf{MAX}(S', \succsim), \mathbf{R}) = C(S')$. Since $z^y \succ y$, it is also clear that $y \notin C(S')$. To summarize, in all contingencies, we are able to find a finite menu $S' \subseteq S$ with $z \in C(S')$ and $y \in S' \backslash C(S')$, as demanded by part (i) of (H6).

Suppose now that z and y both belong to $C(S)$. Since $C(S)$ is an $\mathbf{R}$-cycle, we can find finitely many points $z_0, z_1, ..., z_n$ in $C(S)$ such that $y \in \{z_0, ..., z_n\}$, and $z = z_0\ \mathbf{R}\ z_1\ \mathbf{R} \cdots \mathbf{R}\ z_n = z$. Then, with $S' := \{z_0, ..., z_n\}$, we have $S' = \mathbf{MAX}(S', \succsim)$ because $S' \subseteq C(S) \subseteq \mathbf{MAX}(S, \succsim)$. Moreover, by construction, x tran$(\mathbf{R}|_{S'})$ x' for any $x, x' \in S'$, which implies $C(S') = S'$ by Lemma 3.1. In particular, z and y also belong to $C(S')$, as we sought. Conclusion: C also satisfies part (ii) of (H6).

Since $\mathbf{R}$ is complete, open lower-semicontinuity of this binary relation implies that the set $\{y \in X : y\ \mathbf{R}\ x\}$ is closed for every $x \in X$. Hence, for any convergent sequence (y_n) in X with $y_n\ \mathbf{R}\ x$ for every $n \in \mathbb{N}$, we also have $\lim y_n\ \mathbf{R}\ x$. This implies (H7) because, by rationalization, $y\ \mathbf{R}\ x$ is equivalent to saying $y \in C\{x, y\}$ for any $x, y \in X$.

(a) $\Rightarrow$ (b). Let C be a choice correspondence on $\mathfrak{X}_k$ that satisfies (H1)-(H3) and (H5)-(H7). Define a binary relation $\succcurlyeq$ on X as

$$x \succcurlyeq y \quad \text{iff} \quad x = y \text{ or } x \succ_C y.$$

Here, $x \succ_C y$ means that $y \notin C(S)$ for any $S \in \mathfrak{X}_k$ with $x \in S$. In the main text, we had defined an analogous dominance relation that focuses on finite menus in $\mathfrak{X}$. From (H6), it easily follows that the two definitions are equivalent. That is, $x \succ_C y$ iff $y \notin C(S)$ for any $S \in \mathfrak{X}$ with $x \in S$. Moreover, $C(S) \subseteq \mathbf{MAX}(S, \succcurlyeq)$ for each $S \in \mathfrak{X}_k$ because $y \succ_C x$ does not hold for any $x \in C(S)$ and $y \in S$.

The revealed preference $\mathbf{R}$ is defined as usual: $x\ \mathbf{R}\ y$ iff $x \in C\{x, y\}$. From the proof of Theorem 5.2 in the main text, we know that $(\succcurlyeq, \mathbf{R})$ is a preference structure on X. Indeed, the related arguments in that proof do not utilize Arrow's Undominated Choice Axiom.

To obtain an OLSC core preference, we define a further binary relation $\succcurlyeq^o$ as follows: For any $x, y \in X$,

$$x \succcurlyeq^o y \quad \text{iff} \quad x = y \text{ or } x \succ_C U \text{ for some open set } U \subseteq X \text{ with } y \in U.$$

Clearly, the relation $\succcurlyeq^o$ is antisymmetric. Furthermore, for any $x, y \in X$, we have $x \succ^o y$ iff $x \succ U$ for an open neighborhood U of y.

We proceed with a series of claims.

Claim 1. $(\succcurlyeq^o, \mathbf{R})$ *is an OLSC preference structure.*

Proof of Claim 1. Let us first show that $\succcurlyeq^o$ is transitive. Take any $x, y, z \in X$ with $x \succcurlyeq^o y \succcurlyeq^o z$. If any of these alternatives are the same, we trivially have $x \succcurlyeq^o z$. So, suppose $x \succ^o y \succ^o z$. Then $x \succ y$, and there exists an open neighborhood U of z such that $y \succ U$. Since $\succ$ is transitive, it then follows that $x \succ U$, which implies $x \succ^o z$, as we sought.

Fix an $x \in X$. We claim that $x_{\downarrow}(\succ^o)$ is an open set. Pick any $y \in x_{\downarrow}(\succ^o)$ so that $x \succ^o y$, or equivalently, $x \succ V$ for an open neighborhood V of z. From the definition of $\succ^o$, it immediately follows that $x \succ^o z$ for every $z \in V$, which means $x \succ^o V$. So, the open set V is contained in $x_{\downarrow}(\succ^o)$. Since y is an arbitrarily selected point in $x_{\downarrow}(\succ^o)$, it follows that the latter is open set, as we claimed. Since x is also an arbitrary point, we conclude that $\succcurlyeq^o$ is OLSC.

By (H7), the set $x^{\uparrow}(\mathbf{R})$ is closed for each $x \in X$. Since $\mathbf{R}$ is complete, this is equivalent to saying that $\mathbf{R}$ is OLSC.

As noted earlier, $(\succcurlyeq, \mathbf{R})$ is a preference structure, which means that $\mathbf{R}$ is a $\succcurlyeq$-transitive extension of $\succcurlyeq$. Since $\succcurlyeq^o$ and $\succ^o$ are subrelations of $\succcurlyeq$ and $\succ$, respectively, it obviously follows that $\mathbf{R}$ is also a $\succcurlyeq^o$-transitive extension of $\succcurlyeq^o$. Thus, $(\succcurlyeq^o, \mathbf{R})$ is a preference structure as well. $\|$

The proof of the following claim builds upon (H5) in place of Arrow's Undominated Choice Axiom.

Claim 2. For any nonempty, finite $T \subseteq X$, *we have* $C(T) = \bigcirc(\mathbf{MAX}(T, \succcurlyeq), \mathbf{R})$.

Proof of Claim 2. When $|T| = 1$ the conclusion of the claim holds trivially. Inductively, fix a natural number n, and suppose that $C(S) = \bigcirc(\mathbf{MAX}(S, \succcurlyeq), \mathbf{R})$ for any nonempty $S \subseteq X$ with $|S| \leq n$. Consider a set $T \subseteq X$ with $|T| = n + 1$.

Assume first that $\mathbf{MAX}(T, \succcurlyeq)$ is a proper subset of T. Then, with $S := \mathbf{MAX}(T, \succcurlyeq)$, the induction hypothesis implies

$$C(\mathbf{MAX}(T, \succcurlyeq)) = C(S) = \bigcirc(\mathbf{MAX}(S, \succcurlyeq), \mathbf{R}) = \bigcirc(\mathbf{MAX}(T, \succcurlyeq), \mathbf{R}), \tag{4}$$

where the last equality follows from the definition of the $\mathbf{MAX}(\cdot, \succcurlyeq)$ operator. Recall also that $C(T) \subseteq \mathbf{MAX}(T, \succcurlyeq)$. Thus, (H2) implies $C(T) = C(\mathbf{MAX}(T, \succcurlyeq))$. Then equation (4) yields $C(T) = \bigcirc(\mathbf{MAX}(T, \succcurlyeq), \mathbf{R})$, as we sought.

Suppose now that $\mathbf{MAX}(T, \succcurlyeq) = T$. Pick any $\mathbf{R}$-highset A in T. By definition of an $\mathbf{R}$-highset, we have $C\{x, y\} = \{x\}$ for every $x \in A$ and $y \in T \backslash A$. So, (H1) implies $C(T) \subseteq A$. Thus, every $\mathbf{R}$-highset in T contains $C(T)$.

In the remainder of the proof, we show that $C(T)$ itself is an $\mathbf{R}$-highset in T. From the definition of a top cycle, it then follows that $C(T) = \bigcirc(T, \mathbf{R})$. Since the latter set equals $\bigcirc(\mathbf{MAX}(T, \succcurlyeq), \mathbf{R})$, this completes the proof by induction on the cardinality of the set T.

To this end, we need to establish the following property, along the lines of Arrow's Undominated Choice Axiom: For any pair of sets S and T',

$$S \subseteq T' \subseteq T \text{ and } S \cap C(T') \neq \varnothing \quad \Rightarrow \quad C(S) \subseteq C(T'). \tag{5}$$

Pick a pair of sets S and T' with $S \subseteq T' \subseteq T$ and $S \cap C(T') \neq \varnothing$. We shall assume $S \neq T'$, for otherwise it is trivially true that $C(S) \subseteq C(T')$. Moreover, $S \neq T'$ implies $|S| \leq n$ because $|T'| \leq |T| = n + 1$. Thus, the induction hypothesis applies to the set S.

As a first step, let us assume $|T' \backslash S| = 1$, so that $T' = S \cup \{x\}$ for some $x \in T'$. Pick a point $y \in C(S)$ and a further point $z \in S \cap C(T')$. Our next task is to show that $y \in C(T')$. There are two subcases to consider.

Case 1: $z \in C(S)$. The induction hypothesis and Lemma 3.1 in the main text jointly imply that $C(S)$ is an $\mathbf{R}$-cycle. Thus, in Case 1, there exist finitely many points $y^0, y^1, ..., y^\ell$ in $C(S)$ such that $y = y^0 \ \mathbf{R} \ y^1 \ \mathbf{R} \cdots \mathbf{R} \ y^\ell = z$. By definition of $\mathbf{R}$, the latter means $y^i \in C\{y^i, y^{i+1}\}$ for $i \in \{0, ..., \ell - 1\}$. Moreover, since $\mathbf{MAX}(T, \succcurlyeq) = T$, (H5)—or the weaker property (2)—implies that each element of T is $\blacktriangleright_C$-maximal in this set. In particular, for any $i \in \{0, ..., \ell-1\}$ the point y^i is not $\blacktriangleright_C$-dominated by x. But then $y^{\ell-1} \in C(S) \cap C\{y^{\ell-1}, z\}$ and $z \in C(T') = C(S \cup \{x\})$ jointly imply $y^{\ell-1} \in C(T')$. Indeed, if we had $y^{\ell-1} \notin C(T')$, it would follow that $x \blacktriangleright_C y^{\ell-1}$ by definition of $\blacktriangleright_C$. Similarly, $y^{\ell-2} \in C(S) \cap C\{y^{\ell-2}, y^{\ell-1}\}$ and $y^{\ell-1} \in C(T')$ jointly imply $y^{\ell-2} \in C(T')$ whenever $\ell - 1 > 0$. Proceeding in this fashion, we conclude that $y = y^0 \in C(T')$.

Case 2: $z \notin C(S)$. Note that $C(S)$ is an $\mathbf{R}$-highset in $\mathbf{MAX}(S, \succcurlyeq)$ by the induction hypothesis, while S equals $\mathbf{MAX}(S, \succcurlyeq)$ because $S \subseteq T = \mathbf{MAX}(T, \succcurlyeq)$. Thus, in Case 2, we have $y \ \mathbf{R}^> \ z$, which implies $y \in C(S) \cap C\{y, z\}$. Since $z \in C(T') = C(S \cup \{x\})$, it then follows that $y \in C(T')$. Otherwise, we would have $x \blacktriangleright_C y$, whereas all elements of T are $\blacktriangleright_C$-maximal, as noted in the previous paragraph.

So, in both cases, $y \in C(T')$, as we sought. Since y is arbitrarily selected from $C(S)$, we thereby showed that $S \subseteq T' \subseteq T$, $S \cap C(T') \neq \varnothing$ and $|T' \backslash S| \leq 1$ jointly imply $C(S) \subseteq C(T')$. Inductively, fix an integer $m \geq 1$ and suppose the following holds for any pair of sets S and T':

$$\left|T' \backslash S\right| \leq m, S \subseteq T' \subseteq T \text{ and } S \cap C(T') \neq \varnothing \quad \Rightarrow \quad C(S) \subseteq C(T'). \tag{6}$$

To complete the proof of (5), let us now pick a pair of sets S and T' with $|T' \backslash S| = m + 1$, $S \subseteq T' \subseteq T$ and $S \cap C(T') \neq \varnothing$. For each $x \in T' \backslash S$, define a further set $S_x := S \cup \{x\}$. We claim that

$$C(S_x) \neq \{x\} \quad \text{for some } x \in T' \backslash S. \tag{7}$$

For any $x \in T' \backslash S$, we have $|T' \backslash S_x| = m \geq 1$, and $|T'| \leq n + 1$, which jointly imply $|S_x| \leq n$. Thus, $C(S_x) = \bigcirc(\mathbf{MAX}(S_x, \succcurlyeq), \mathbf{R})$ by our main induction hypothesis. Since each element of T is $\succcurlyeq$-maximal, we also have $\mathbf{MAX}(S_x, \succcurlyeq) = S_x$. Hence, if $C(S_x) = \{x\}$, then $\{x\}$ equals $\bigcirc(S_x, \mathbf{R})$, which is an $\mathbf{R}$-highset in S_x. Consequently, we have $x \ \mathbf{R}^> \ y$, i.e., $C\{x, y\} = \{x\}$ for any $y \in S$. So, if $C(S_x) = \{x\}$ for each $x \in T' \backslash S$, then $C\{x, y\} = \{x\}$ for every $x \in T' \backslash S$ and $y \in S$. In this case, (H1) would imply $C(T') \subseteq T' \backslash S$, contradicting the hypothesis $S \cap C(T') \neq \varnothing$. This proves (7).

By (7), we can pick an $x \in T' \backslash S$ with $C(S_x) \neq \{x\}$, so that $S \cap C(S_x) \neq \emptyset$. Since $|S_x \backslash S| = 1 \leq m$, from the property (6) it follows that $C(S) \subseteq C(S_x)$. Moreover, $S_x \cap C(T')$ is also nonempty because it contains the nonempty set $S \cap C(T')$. As $|T' \backslash S_x| = m$, we can apply (6) one more time to conclude $C(S_x) \subseteq C(T')$. That is, $C(S) \subseteq C(S_x) \subseteq C(T')$. This shows that (5) holds irrespective of the cardinality of the set $T' \backslash S$.

Finally, pick any $z \in C(T)$ and $y \in T$ such that $y \ \mathbf{R} \ z$. Since $\{z, y\} \cap C(T)$ is nonempty, with $S := \{z, y\}$ and $T' := T$, property (5) implies $C(S) \subseteq C(T)$. As $y \in C(S)$, it follows that $y \in C(T)$. In other words, we have $y \in C(T)$ for each $y \in T$ such that $y \ \mathbf{R} \ z$ for some $z \in C(T)$. Thus, $C(T)$ is an $\mathbf{R}$-highset in T, as we claimed. $\|$

Claim 3. Let $S \subseteq X$ be a nonempty finite set, and suppose that for some $z, y \in S$, we have $z \in C(S)$, $y \in S \backslash C(S)$ and $y \ \mathbf{R} \ z$. Then there exists an $x \in S$ such that $x \blacktriangleright_C y$.

Proof of Claim 3. By Claim 2, $C(S)$ is an $\mathbf{R}$-highset in $\mathbf{MAX}(S, \succcurlyeq)$. Thus, $z \in C(S)$, $y \in S \backslash C(S)$ and $y \ \mathbf{R} \ z$ jointly imply $y \notin \mathbf{MAX}(S, \succcurlyeq)$. So, there exists a point $x' \in S$ such that $x' \succ y$. Since S is finite, we can assume that $x' \in \mathbf{MAX}(S, \succcurlyeq)$. As $z \in C(S) = \bigcirc(\mathbf{MAX}(S, \succcurlyeq), \mathbf{R})$, Lemma 3.1 in the main text implies z tran$(\mathbf{R}|_{\mathbf{MAX}(S,\succcurlyeq)})$ x', meaning that there exist some $z_0, z_1, ..., z_n$ in $\mathbf{MAX}(S, \succcurlyeq)$ such that $z = z_0 \; \mathbf{R} \; z_1 \; \mathbf{R} \cdots \mathbf{R} \; z_n = x'$.

Let i denote the smallest index $j \in \{0, 1, ..., n\}$ such that $z_j \succ y$. (Such j exists since $z_n = x' \succ y$.) Note that $z_0 \succ y$ does not hold because we have $y \; \mathbf{R} \; z_0$, and $\succ$ is a subrelation of $\mathbf{R}^{>}$. Thus, z_i is distinct from z_0.

Define a set $S' := \{z_0, z_1, ..., z_i\} \cup \{y\}$. Then $C(S') = \bigcirc(\mathbf{MAX}(S', \succcurlyeq), \mathbf{R})$ by Claim 2. We shall now show that

$$z_0 \in C(S'),\ y \notin C(S'),\ \text{and}\ y \in C(S' \backslash \{z_i\}). \tag{8}$$

First note that $\{z_0, ..., z_i\} \subseteq \mathbf{MAX}(S', \succcurlyeq)$ because $\mathbf{MAX}(S, \succcurlyeq) \cap S' \subseteq \mathbf{MAX}(S', \succcurlyeq)$. As $z_i \succ y$, we actually have $\{z_0, ..., z_i\} = \mathbf{MAX}(S', \succcurlyeq)$. This implies $y \notin C(S')$, for $C(S')$ is a subset of $\mathbf{MAX}(S', \succcurlyeq)$. Moreover, by construction, z_0 tran$(\mathbf{R}|_{\mathbf{MAX}(S',\succcurlyeq)})$ z_j for $1 \leq j \leq i$. From Lemma 3.1 in the main text, it follows that $z_0 \in C(S')$. It remains to show that $y \in C(S' \backslash \{z_i\})$. To this end, let $S'' := S' \backslash \{z_i\}$. Then $y \in \mathbf{MAX}(S'', \succcurlyeq)$ by definition of the index i. This also implies $S'' = \mathbf{MAX}(S'', \succcurlyeq)$ because all points in S'' that are distinct from y are already $\succcurlyeq$-maximal in the larger set S'. Furthermore, since $y \; \mathbf{R} \; z_0$ and z_0 tran$(\mathbf{R}|_{S''})$ z_j, we have y tran$(\mathbf{R}|_{S''})$ z_j for every $z_j \in S'' \backslash \{y\}$. Thus, Lemma 3.1 in the main text and Claim 2 above imply $y \in C(S'')$, as we sought.

Finally, $y \; \mathbf{R} \; z_0$ means $y \in C\{y, z_0\}$. Thus, by (8), the point $x := z_i$ satisfies $x \blacktriangleright_C y$. ‖

Claim 4. For any $S \in \mathfrak{X}_k$, $C(S)$ is an $\mathbf{R}$-highset in $\mathbf{MAX}(S, \succcurlyeq^o)$.

Proof of Claim 4. Fix an $S \in \mathfrak{X}_k$. Note that $\mathbf{MAX}(S, \succcurlyeq) \subseteq \mathbf{MAX}(S, \succcurlyeq^o)$ because $\succ^o$ is a subrelation of $\succ$. Since $C(S) \subseteq \mathbf{MAX}(S, \succcurlyeq)$, it follows that $C(S)$ is a subset of $\mathbf{MAX}(S, \succcurlyeq^o)$.

Pick any $z \in C(S)$ and $y \in \mathbf{MAX}(S, \succcurlyeq^o) \backslash C(S)$. If we can show that $z \; \mathbf{R}^{>} \; y$, we can immediately conclude that $C(S)$ is an $\mathbf{R}$-highset in $\mathbf{MAX}(S, \succcurlyeq^o)$.

By way of contradiction, suppose $y \; \mathbf{R} \; z$. Part (i) of (H6) postulates a finite set $S' \subseteq S$ with $z \in C(S')$ and $y \in S' \backslash C(S')$. By Claim 3, these properties of S', z and y imply the existence of a point $x \in S'$ such that $x \blacktriangleright_C y$. Then, by (H5), we also have $x \succ_C U$ for an open set U with $y \in U$. This means $x \succ^o y$ by definition of $\succ^o$. Note, however, that x belongs to S', which is a subset of S, contradicting the assumption $y \in \mathbf{MAX}(S, \succcurlyeq^o)$. Hence, $z \; \mathbf{R}^{>} \; y$, as we sought. ‖

Claim 5. For any $T \in \mathfrak{X}_k$ and any $\mathbf{R}$-highset A in $\mathbf{MAX}(T, \succcurlyeq^o)$, we have $C(T) \subseteq A$.

Proof of Claim 5. Take a $T \in \mathfrak{X}_k$ and an $\mathbf{R}$-highset A in $\mathbf{MAX}(T, \succcurlyeq^o)$. By Claim 4, $C(T)$ is also an $\mathbf{R}$-highset in $\mathbf{MAX}(T, \succcurlyeq^o)$. Since $\mathbf{R}$-highsets are nested, it follows that either $C(T) \subseteq A$ or $A \subseteq C(T)$. As our purpose is to prove the former inclusion, without loss of generality we shall assume $A \subseteq C(T)$. It is also worth noting that A is nonempty because it is an $\mathbf{R}$-highset.

Let $S := \mathbf{MAX}(T, \succcurlyeq^o)$. This is a nonempty, compact set by Lemma 1 and Claim 1. Moreover, (H2) implies $C(T) = C(S)$ because we have $C(T) \subseteq S \subseteq T$.

We shall complete the proof by showing that $C(S) \subseteq A$. Pick any $y \in C(S)$. Since A is nonempty, there also exists a $z \in A$. Since $A \subseteq C(T) = C(S)$, the point z belongs to $C(S)$ as well. By part (ii) of (H6), then, there exists a finite set $S' \subseteq S$ such that $z, y \in C(S')$.

Claim 2 implies that $C(S')$ equals $\bigcirc(\mathbf{MAX}(S', \succcurlyeq), \mathbf{R})$, which is an $\mathbf{R}$-cycle by Lemma 3.1 in the main text. Thus, there exist some $y_0, y_1, ..., y_n$ in S' such that $y = y_0 \; \mathbf{R} \; y_1 \; \mathbf{R} \cdots \mathbf{R} \; y_n = z$. Moreover, $y_{n-1} \; \mathbf{R} \; y_n$ and $y_n = z \in A$ jointly imply $y_{n-1} \in A$ because A is an $\mathbf{R}$-highset in S and $y_{n-1} \in S' \subseteq S$. Similarly, $y_{n-2} \; \mathbf{R} \; y_{n-1}$ and $y_{n-1} \in A$ jointly imply $y_{n-2} \in A$ provided that $n - 1 > 0$. Proceeding in this fashion, we find that $y = y_0 \in A$. Since y was arbitrarily selected from $C(S)$, we conclude that $C(S) \subseteq A$. ‖

Claims 4 and 5 jointly imply $C(S) = \bigcirc(\mathbf{MAX}(S, \succcurlyeq^o), \mathbf{R})$ for any $S \in \mathfrak{X}_k$. Hence, the choice correspondence C is rationalized by the OLSC preference structure $(\succcurlyeq^o, \mathbf{R})$. It remains to show that $\succcurlyeq^o$ is weakly spacious.

Take any convergent sequence (x_n) in X such that $x_{n+1} \succ^o x_n$ for each $n \in \mathbb{N}$. Set $x := \lim x_n$. We need to show that $x \succ^o x_1$, or equivalently, $x \succ U$ for an open neighborhood U of x_1.

As $x_2 \succ^o x_1$, there exists an open neighborhood U of x_1 such that $x_2 \succ U$. We claim that $x \succ U$ also holds. Pick any $y \in U$, and any $S \in \mathfrak{X}_k$ with $x \in S$. Let $S' := \{x_n : n \in \mathbb{N}\} \cup \{x, y\}$ and $T := S \cup S'$. Note that T is compact as a union of two compact sets. Moreover, for any $n \in \mathbb{N}$, the point x_n does not belong to $C(T)$ because $x_{n+1} \succ^o x_n$ implies $x_{n+1} \succ_C x_n$ by definition of $\succ^o$. Similarly, y does not belong to $C(T)$ because $x_2 \succ y$ implies $x_2 \succ_C y$. So, x is the only element of S' that can belong to $C(T)$. Since $x \in S$, it follows that $C(T) \subseteq S$, while (H2) implies $C(T) = C(S)$. But then y does not belong to $C(S)$ either.

We have thereby shown that $y \notin C(S)$ for any $y \in U$, and any $S \in \mathfrak{X}_k$ with $x \in S$. This simply means $x \succ_C U$, or equivalently, $x \succ U$, as we sought. So, $\succcurlyeq^o$ is weakly spacious, which completes the proof.

(b) $\Leftrightarrow$ (c). That (b) implies (c) is obvious. For the converse, suppose that C is rationalized on $\mathfrak{X}_k$ by an OLSC weak preference structure $(\succsim, \mathbf{R})$ on X. Assume further that $\succsim$ is weakly spacious. Define a binary relation $\mathbf{R}_{\succsim}$ on X as $x \ \mathbf{R}_{\succsim} \ y$ iff

$$\text{either } x \succsim y \text{ or } \left[(x,y) \in \mathrm{Inc}(\succsim) \text{ and } x \ \mathbf{R} \ \mathbf{y}\right].$$

By Lemma A.2 in the main text, $(\succsim, \mathbf{R}_{\succsim})$ is a preference structure. It is also clear that for any $S \in \mathfrak{X}_k$ and $x, y \in \mathbf{MAX}(S, \succsim)$, we have $x \ \mathbf{R} \ \mathbf{y}$ iff $x \ \mathbf{R}_{\succsim} \ y$. Thus, for each $S \in \mathfrak{X}_k$ we have $\bigcirc(\mathbf{MAX}(S, \succsim), \mathbf{R}) = \bigcirc(\mathbf{MAX}(S, \succsim), \mathbf{R}_{\succsim})$, which implies that the preference structure $(\succsim, \mathbf{R}_{\succsim})$ rationalizes C as well.

It remains to show that $\mathbf{R}_{\succsim}$ is OLSC. As $(\succsim, \mathbf{R}_{\succsim})$ and $(\succsim, \mathbf{R})$ both rationalize C, by definitions of a preference structure and a weak preference structure respectively, for any pair of distinct points $x, y \in X$ we have

$$x \ \mathbf{R}^{>}_{\succsim} \ y \quad \Leftrightarrow \quad \{x\} = C\{x,y\} \quad \Leftrightarrow \quad y \notin \max\left(\{x,y\}, \mathbf{R}\right) \cap \mathbf{MAX}\left(\{x,y\}, \succsim\right).$$

The right hand side of this expression means either $x \ \mathbf{R}^{>} \ y$ or $x \succ y$. It thus follows that $\{y \in X : x \ \mathbf{R}^{>}_{\succsim} \ y\} = \{y \in X : x \ \mathbf{R}^{>} \ y\} \cup \{y \in X : x \succ y\}$ for any $x \in X$. Since $\mathbf{R}$ and $\succsim$ are both OLSC, we conclude that for each $x \in X$ the set $\{y \in X : x \ \mathbf{R}^{>}_{\succsim} \ y\}$ is open as a union of two open sets. So, $\mathbf{R}_{\succsim}$ is also OLSC, as we sought.

Proof of Theorem 2

(c) $\Rightarrow$ (b). Let $\succsim$ be an OLSC and weakly spacious preorder on X such that $C(S) = \mathbf{MAX}(S, \succsim)$ for each $S \in \mathfrak{X}_k$. Set $\mathbf{R} := X \times X$, which is a complete, transitive and OLSC binary relation such that $\bigcirc(S, \mathbf{R}) = S$ for each $S \in \mathfrak{X}_k$. Thus, the transitive weak preference structure $(\succsim, \mathbf{R})$ rationalizes C on $\mathfrak{X}_k$ and satisfies all topological conditions mentioned in statement (b).

(b) $\Rightarrow$ (a). Let $(\succsim, \mathbf{R})$ be an OLSC and transitive weak preference structure that rationalizes C on $\mathfrak{X}_k$. Assume further that $\succsim$ is weakly spacious. Then, by Lemma 2, $\succsim$ also possesses MDP on $\mathfrak{X}_k$. Moreover, just as in the main text, MDP, transitivity of $(\succsim, \mathbf{R})$ and rationalization jointly imply

$$C(S) = \bigcirc(\mathbf{MAX}\left(S, \succsim\right), \mathbf{R}) = \max\left(S, \mathbf{R}\right) \cap \mathbf{MAX}\left(S, \succsim\right) \quad \text{for every } S \in \mathfrak{X}_k.$$

It easily follows that C satisfies the Chernoff Axiom because the correspondences $S \to \max(S, \mathbf{R})$ and $S \to \mathbf{MAX}(S, \mathbf{R})$ both satisfy this axiom. Finally, note that all remaining axioms follow from Theorem 1.

(a) $\Rightarrow$ (c). Suppose that C satisfies (H2), (H3') and (H5)-(H7). Then (H1) also holds as a consequence of (H3'). By Theorem 1, there exists an OLSC preference structure $(\succsim, \mathbf{R})$, with a weakly spacious core preference $\succsim$, such that $C(S) = \bigcirc(\mathbf{MAX}(S, \succsim), \mathbf{R})$ for each $S \in \mathfrak{X}_k$. Let us define $\succsim$ as in the proof of Theorem 1 so that $\succ$ equals $\succ^o$, which means $x \succ y$ iff $x \succ_C U$ for some open set $U \subseteq X$ with $y \in U$.

We claim that $\mathbf{R}^>$ is contained in $\succ$. Pick any $x, y \in X$ such that $x \succ y$ is false. Then $x \succ_C U$ is false for any open neighborhood U of y. Since X is a metric space, it clearly follows that there exists a sequence (y_n) such that $y = \lim y_n$, and $x \succ_C y_n$ is false for each $n \in \mathbb{N}$. In turn, if $x \succ_C y_n$ is false, there exists a menu $S_n \in \mathfrak{X}_k$ with $y_n \in C(S_n)$ and $x \in S_n$. Thus, the Chernoff axiom implies $y_n \in C\{x, y_n\}$ for each n, while (H7) entails $y \in C\{x, y\}$. But then we must also have $y \mathbf{R} x$ because $(\succsim, \mathbf{R})$ is a preference structure that rationalizes C. To summarize, if $x \succ y$ is false, then $x \mathbf{R}^> y$ is also false, proving that $\mathbf{R}^>$ is a subrelation of $\succ$.

Fix an $S \in \mathfrak{X}_k$. As we have just seen, for any $x, y \in S$ with $x \mathbf{R}^> y$, we also have $x \succ y$, and hence, $y \notin \mathbf{MAX}(S, \succsim)$. It follows that $x \mathbf{R}^= y$ for each $x, y \in \mathbf{MAX}(S, \succsim)$, which implies $\bigcirc(\mathbf{MAX}(S, \succsim), \mathbf{R}) = \mathbf{MAX}(S, \succsim)$. Thus, $C(S) = \mathbf{MAX}(S, \succsim)$ for each $S \in \mathfrak{X}_k$, as we sought.

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