Non-Existence of Continuous Choice *Functions*[*]

Hiroki Nishimura[†] Efe A. Ok[‡]

June 21, 2013

Abstract

Let X be a compact, or path-connected, metric space whose topological dimension is at least 2. We show that there does not exist a continuous choice function (i.e., single-valued choice correspondence) defined on the collection of all finite feasible sets in X. Not to be void of content, therefore, a revealed preference theory in the context of most infinite consumption spaces must either relinquish the fundamental continuity property or allow for *multi*-valued choice correspondences.

JEL Classification: D11, D81.
Keywords: Choice Functions, Rationality, Continuity.

1 INTRODUCTION

The primitives of revealed preference analysis for an individual are a universal set X (of choice alternatives), a collection $\mathcal{A}$ of nonempty subsets of X (to serve as the collection of all choice problems that the agent may potentially encounter), and a correspondence c mapping each element of $\mathcal{A}$ to a nonempty subset of that element (which is interpreted as the choice correspondence of the individual that tells us what she may choose in any given choice problem). In such a context, taking the choice correspondence c as single-valued (so that c is a *function* from $\mathcal{A}$ into X) often simplifies the analyses considerably. Especially in the recent body of research on boundedly rational choice theory, this modeling strategy is widely adopted.

It is, however, important to note that taking choice functions as primitives of analysis have behavioral implications. Indeed, positing that in every choice problem one can identify a unique alternative to choose assumes away some potentially interesting traits such as indifference and indecisiveness, thereby limiting the foundational nature of the involved model. (To wit, a preference relation over risky alternatives deduced from a choice *function* says that no nondegenerate lottery has a certainty equivalent.) This is, of course, not a novel point, but one that has been debated in the folklore extensively. In the present note, we would like to contribute to this debate by making a formal observation about single-valued choice correspondences.

[*]We thank Gabriel Carroll, Lars Ehlers, Matt Jackson, Marco Mariotti and Ariel Rubinstein for insightful discussions about the content of this paper.

[†]Department of Economics, New York University, 19th West 4th Street, New York, NY 10012. E-mail: hiroki@nyu.edu

[‡][*Corresponding Author*] Department of Economics, New York University, 19th West 4th Street, New York, NY 10012. E-mail: efe.ok@nyu.edu

In most economic frameworks, the universal sets of alternatives are infinite, and they are modeled as metric spaces. (In demand theory, for instance, this space is often taken as $\mathbb{R}^n_+$, in risk theory as the convex hull of a collection of points in a normed linear space, and in the theory of intertemporal choice as some suitable sequence space.) Revealed preference analyses in such contexts posit some form of continuity (such as the closed graph property) on the part of the choice correspondences. Not only is this reasonable, but it is often necessary for deriving utility functions, or at least for ensuring the existence of extrema with respect to preference (or other types of binary) relations induced from choice behavior. The goal of this note is to show that unless the alternative space has a rather esoteric structure, or it is topologically equivalent to an interval in the real line, no choice function can be continuous on a domain that contains all finite choice problems. For example, if X is any (nontrivial) normed linear space and $\mathcal{A}$ is the collection of all nonempty finite subsets of X, then there exists a continuous choice function on $\mathcal{A}$ (with $\mathcal{A}$ being metrized by the standard Hausdorff metric) if, and only if, X is homeomorphic to $\mathbb{R}$. In particular, there is no continuous choice function on the set of all nonempty finite subsets of $\mathbb{R}^n$ for any $n \geq 2$. In fact, so long as X is a metric space that contains a compact (or path-connected) subspace that is not homeomorphic to a subset of the real line, no choice function can be continuous on the set of all finite (or even doubleton) subsets of X.

It thus appears that there is an intrinsic clash between the properties of continuity and single-valuedness for choice correspondences. In idealized situations where one works with a complete set of single-observation choice data for finite sets, the data will often take a discontinuous form. In a nutshell, and in the words of one of the referees of this paper, "any (choice) theory that one comes up with to explain those data "exactly" (that is, without admitting that theory could also have generated a different set of single observations) will have to imply discontinuous behavior," unless the alternative space has a particularly simple structure.

In what follows, Section 2 introduces some preliminaries and Section 3 contains some examples, the statement of our main result and an application of it to the theory of incomplete preferences, as well as a brief discussion on the use of choice functions in revealed preference analysis. Section 4 provides a proof for our main result.

2 PRELIMINARIES

2.1 Choice Correspondences. Let X be a metric space. We denote the collection of all nonempty subsets of X that contain at most k elements by $\mathcal{F}_k(X)$. The collection of all nonempty finite subsets of X is then denoted by $\mathcal{F}(X)$, and that of all nonempty compact subsets of X by $\mathbf{k}(X)$. Throughout the present paper, we regard $\mathcal{F}_k(X)$ (for any positive integer k), $\mathcal{F}(X)$ and $\mathbf{k}(X)$ as metric spaces relative to the Hausdorff metric.[1]

As usual, we interpret X as a universal set of choice alternatives and any member of $\mathcal{F}(X)$ as a feasible set (choice problem) that an agent may face.[2] By a *choice correspondence* on $\mathcal{F}(X)$, we mean any map $c : \mathcal{F}(X) \to 2^X$ such that $\emptyset \neq c(S) \subseteq S$ for each $S \in \mathcal{F}(X)$. A *choice function* on $\mathcal{F}(X)$ is a choice correspondence c on $\mathcal{F}(X)$ with $|c(S)| = 1$ for every $S \in \mathcal{F}(X)$. For a choice function c on $\mathcal{F}(X)$, we identify the set $c(S)$ with the unique element that it contains for each $S \in \mathcal{F}(X)$, thereby treating c as a function from $\mathcal{F}(X)$ into X.

2.2 Continuity of Choice Correspondences. Given any metric space X and any choice correspondence c on $\mathcal{F}(X)$, we may view c as a self-map on the metric space $\mathcal{F}(X)$, and ask for its continuity in the usual (topological) sense. This is, in general, too demanding; even those choice correspondences

[1] The Hausdorff metric and convergence with respect to this metric are denoted by d^{H} and $\to^{\mathrm{H}}$, respectively. We recall that $d^{\mathrm{H}}(S,T)$ is a uniform metric in the sense that it equals $\sup\{|d(x,S) - d(x,T)| : x \in X\}$, where d is the metric of the underlying space X.

[2] While this formulation is fairly standard, many revealed preference analyses are conducted by taking the collection $\mathbf{k}(X)$ as the set of all feasible choice problems. Every result that we report in this note is valid on this larger domain as well. More generally, in any of our results, one can replace $\mathcal{F}(X)$ by any collection $\mathcal{D}$ with $\mathcal{F}(X) \subseteq \mathcal{D} \subseteq \mathbf{k}(X)$. As our findings are of "non-existence" type, keeping the domain of choice functions smaller makes these results formally stronger. (In fact, we can do better in this regard; see Remark 3.1 below.)

that are characterized by the maximization of a continuous utility function may fail to satisfy this property. For this reason, the most commonly adopted notion of continuity for choice correspondences is that of *upper hemicontinuity.* (The Berge Maximum Theorem ensures that a choice correspondence on $\mathcal{F}(X)$ of the form $S \mapsto \arg\max\{u(x) : x \in S\}$ is upper hemicontinuous for any continuous real map u on X.) We will in fact work with a weaker notion than this, namely, with the *closed graph property.* For ease of reference, however, we shall refer to any choice correspondence c on $\mathcal{F}(X)$ with a closed graph as "continuous". Put precisely, we say that c is *continuous* if for every $S, S_1, S_2, ...$ in $\mathcal{F}(X)$ and $x, x_1, x_2, ...$ in X such that $S_m \to^{\mathrm{H}} S$, $x_m \to x$ and $x_m \in c(S_m)$ for each m, we have $x \in c(S)$.[3] In this terminology, a choice function c on $\mathcal{F}(X)$ is continuous iff $\lim c(S_m) = c(\lim S_m)$ for every convergent sequence $(S_m) \in \mathcal{F}(X)^\infty$ such that $(c(S_m))$ converges in X (with the understanding that $\lim S_m$ is taken relative to the Hausdorff metric).

3 MAIN RESULTS

3.1 Non-Existence of Continuous Choice Functions: Examples. Let X be a metric space. Our main query here can be summed up as follows: If there exists a continuous choice function c on $\mathcal{F}(X)$, what sort of properties must X satisfy? The importance of this question depends on its answer. If the existence of such c does not put restrictive conditions on X, then this query would be of technical interest at best. At the opposite end, if this existence condition rules out many types of commodity spaces, then we see that working with continuous choice functions may have undesirable consequences. In what follows, we provide several examples that demonstrate that this is indeed the case.

We begin with a simple observation that generates many concrete illustrations.

Example 3.1. Let S^1 stand for the circle in $\mathbb{R}^2$, and suppose c is a continuous choice function on $\mathcal{F}_2(S^1)$. Take any two diametrically opposed points in S^1, say, x and y. Let $x = c\{x, y\}$ without loss of generality. Now think of continuously rotating the points x and y counterclockwise keeping them diametrically opposed. It is intuitively clear that continuity of c forces it to choose the rotated version of x in every resulting pairwise choice problem, and thus eventually c chooses y in the problem $\{x, y\}$, which is a contradiction.[4] Conclusion: There is no continuous choice function on $\mathcal{F}_2(S^1)$.

Example 3.2. Let X be a metric space that contains a subset T that is homeomorphic to a circle in $\mathbb{R}^2$. If there were a continuous choice function on $\mathcal{F}_2(X)$, there would be one on $\mathcal{F}_2(T)$, contradicting what we have found in Example 3.1. Conclusion: There is no continuous choice function on $\mathcal{F}_2(X)$.

Remark 3.1. In the context of cooperative bargaining, Ok and Zhou (1999) have shown that, given any integer $n \geq 2$, no continuous choice function (that is, single-valued bargaining solution) b over the collection Ω^n of all compact and comprehensive sets in $\mathbb{R}^n_+$ can be strongly Pareto optimal. As simple as it is, the observation we have found in Example 3.3 yields this fact as an immediate corollary. For, suppose b is a continuous map from Ω^n into $\mathbb{R}^n$ such that $b(S) \in S$ for each $S \in \Omega^n$. For any x and y in $\mathbb{R}^n_+$, let $S_{x,y}$ be the set of all nonnegative n-vectors z with either $x \geq z$ or $y \geq z$ (componentwise), and define $c : \mathcal{F}_2(\mathbb{R}^n_+) \to \mathbb{R}^n_+$ by $c\{x, y\} := b(S_{x,y})$. If b is strongly Pareto optimal, that is, $y > b(S)$ for

[3] Viewed as a correspondence from $\mathcal{F}(X)$ into X, the map c is closed-valued in X. Consequently, the closed graph property of c is implied by its upper hemicontinuity. (See, for instance, Ok (2007), Proposition E.3.) It is also easy to see that these two notions are identical for choice *functions* from $\mathcal{F}(X)$ into X.

[4] To provide a formal argument for this, let us identify $\mathbb{R}^2$ with the complex plane so that $S^1 = \{e^{\theta i} : 0 \leq \theta < 2\pi\}$. For every θ in $[0, 2\pi)$, define $A(\theta) := \{e^{\theta i}, e^{(\theta+\pi)i}\}$, and suppose $c(A(0)) = e^{\pi i}$ without loss of generality. Define $\theta^* := \sup\{\theta \in [0, \pi] : c(A(\theta)) = e^{(\theta+\pi)i}\}$, and note that, as $\theta \mapsto c(A(\theta))$ is a continuous function on $[0, 2\pi)$, we actually have

$$\theta^* = \max\{\theta \in [0, \pi] : c(A(\theta)) = e^{(\theta+\pi)i}\}.$$

Besides, $\theta^* < \pi$ because $A(\pi) = A(0)$, so $e^{\pi i} = c(A(\pi)) \neq e^{2\pi i}$. It follows that $c(A(\theta^*)) = e^{(\theta^*+\pi)i}$ and $c(A(\theta^* + \varepsilon)) = e^{(\theta^*+\varepsilon)i}$ for arbitrarily small $\varepsilon > 0$, contradicting c being continuous.

no $y \in S$ and $S \in \Omega^n$, then c is a choice function on $\mathcal{F}_2(\mathbb{R}^n_+)$. Moreover, continuity of b implies that of c, but Example 3.2 shows that no such choice function exists.

Example 3.3. For any positive integer n, it is plain that a circle can be (topologically) embedded in $\mathbb{R}^n$ iff $n \geq 2$. In view of the previous exercise, therefore, we conclude: There exists a continuous choice function on $\mathcal{F}_2(\mathbb{R}^n)$ iff $n = 1$.[5]

Example 3.4. Let X be a normed linear space. Then, there exists a continuous choice function on $\mathcal{F}(X)$ iff $\dim X \leq 1$.[6] The proof is identical to that for the previous example. (The "if" part of the result follows from the fact that a normed linear space can be homeomorphic to an interval in $\mathbb{R}$ iff it has (linear) dimension 1.)

Example 3.5. Let Ω be a metric space, and $\triangle(\Omega)$ the set of all Borel probability measures on Ω. We view $\triangle(\Omega)$ as metrized in such a way that its topology coincides with the topology of weak convergence. Relative to this topology, $\lambda \mapsto \lambda p + (1-\lambda)q$ is a continuous injection from $[0,1]$ into $\triangle(\Omega)$ for any p and q in $\triangle(\Omega)$. It follows that, when $|\Omega| \geq 3$, $\triangle(\Omega)$ contains a set that is homeomorphic to a triangle, and hence to a circle, in $\mathbb{R}^2$. It thus follows from Example 3.2 that no choice function can be continuous on $\mathcal{F}(\triangle(\Omega))$, except in the trivial case in which Ω consists of *at most* two points.

The previous examples suggest that continuity is too demanding for choice functions on the finite subsets of a metric space, unless that metric space has a very special structure. These examples are based on the fact that there is no continuous choice function on $\mathcal{F}_2(X)$ for any metric space in which one can embed a circle (or any closed simple loop). However, "embeddability of a circle," while a useful criterion, does not really highlight the (topological) source of the difficulty with continuous choice functions. The following examples demonstrate that continuity of a choice function may well be too demanding even in the context of alternative spaces that do not meet this criterion.

Example 3.6. Fix any positive integer $n \geq 3$, and let X be the set of all nonnegative real n-vectors at most one term of which is nonzero; we view X as a metric subspace of $\mathbb{R}^n$. (Such outcome spaces arise, for instance, in the theory of time preferences (cf. Benoit and Ok (2007), or in (finite-horizon) Stahl-type bargaining games.) Then, one cannot embed a circle in X, but it is still true that there is no continuous choice function on $\mathcal{F}_2(X)$.

Example 3.7. Let X be a metric space, and recall that an *arc* in X is a subset of X which is homeomorphic to $[0,1]$. In what follows, we denote an arc in X with endpoints v and w as $a(v,w)$. (Notice that v and w must be distinct here, because an arc cannot be homeomorphic to a circle.) A subset G of X is said to be a (topological) *graph* in X, if $G = \bigcup E$ for some nonempty finite collection E of arcs in X such that any two distinct elements of E intersect (if at all) only at their endpoints. (The collection V_E of all end points of the elements of E is called the set of *vertices* of G. The *degree* of a vertex $v \in V_E$ is defined as the number of all arcs in E one of whose endpoints is v.) We say that such a graph G is *circuit-free* if there do not exist distinct vertices $v_1, ..., v_k$ (with $k \geq 2$) of it such that E contains arcs of the form $a(v_1, v_2), ..., a(v_{k-1}, v_k)$ and $a(v_k, v_1)$. (For instance, if $\bigcup E$ is path-connected, $|E| = n - 1$ and $|V_E| = n$ for some integer $n \geq 2$, then the graph at hand would be circuit-free.)

Let G be a circuit-free graph in X. Then, it is impossible to embed a circle in G, so it is not evident from the analysis of the previous examples if there is a continuous choice function on the collection of,

[5] More generally, if X is a subset of $\mathbb{R}^n$ with nonempty interior, then there exists a continuous choice function on $\mathcal{F}_2(X)$ iff $n = 1$.

[6] More generally, this result is valid whenever X is a locally convex metric linear space. Even more generally, when X is a connected subset of such a metric linear space with nonempty relative interior, then there exists a continuous choice function on $\mathcal{F}(X)$ iff the affine hull of X is 1-dimensional.

say, all finite subsets of G. In fact, provided that there is at least one vertex of G with degree 3 or more, there is no such choice function.[7]

The non-existence results in Examples 3.6 and 3.7 arise due to the existence of a point in the alternative space removal of which yields at least three locally connected components in arbitrarily small neighborhoods of that point. Our final example shows that this too is only a sufficent, but not necessary, condition for the non-existence of continuous choice functions.

Example 3.8. For any positive integer n, put $I_n := \{\frac{1}{n}\} \times [0,1]$ and set $I_\infty := \{0\} \times [0,1]$. Let X be the set $I_1 \cup \cdots \cup I_\infty$, which we view as a metric subspace of $\mathbb{R}^2$. Clearly, one cannot embed a circle in X, nor is there a point in X removal of which yields three or more locally connected components in arbitrarily small neighborhoods of it. However, it is still true that there is no continuous choice function on $\mathcal{F}_2(X)$.

3.2 Non-Existence of Continuous Choice Functions: Theorems. The examples we have considered so far utilize rather special topological properties of the underlying alternative spaces to obtain non-existence results for continuous choice functions. In fact, all of these examples are special cases of two general observations. It turns out that if a metric space has a compact subspace that is not homeomorphic to a subset of the real line, then no choice function on the finite (or even doubleton) subsets of that space can be continuous. Precisely the same is true for path-connected subspaces as well.

Theorem 3.1.a. Let X be a compact metric space. Then, there is a continuous choice function on $\mathcal{F}(X)$ if and only if X is homeomorphic to a subset of $\mathbb{R}$.

Theorem 3.1.b. Let X be a path-connected metric space. Then, there is a continuous choice function on $\mathcal{F}(X)$ if and only if X is homeomorphic to a real interval.

Under the hypothesis of either path-connectedness or compactness, therefore, working with a continuous choice function on $\mathcal{F}(X)$ is meaningful only for those metric spaces X that are topologically equivalent to a subset of the real line.

Examples 3.6-8. [Continued] The assertions we made in Examples 3.6-8 are immediate consequences of the theorems above. In the context of Example 3.6, all we need is to apply Theorem 3.1.b. Indeed, in that context, there is no continuous choice function on $\mathcal{F}_2(X)$, because X is path-connected, but it is not homeomorphic to an interval (because, for instance, deleting the point $(0,...,0)$ from X yields a space with more than 2 connected components). In the context of Example 3.7, our assertion is a consequence of Theorem 3.1.a. For, any graph G in a metric space is compact, but it cannot be homemorphic to a subset of $\mathbb{R}$ if G has a vertex with degree at least 3. Finally, Theorem 3.1.a also applies to Example 3.8. In that context, there is no continuous choice function on $\mathcal{F}_2(X)$, because X is compact (being a closed subset of $[0,1]^2$), and yet it is not homeomorphic to the union of a countably infinite collection of disjoint intervals.

Theorem 3.1 works with either compactness or path-connectedness. In passing, we note that the simultaneous presence of these properties yields a slightly sharper characterization:

Corollary 3.2. Let X be a compact and connected metric space with $|X| > 1$. Then, there exists a continuous choice function on $\mathcal{F}(X)$ if and only if X is homeomorphic to $[0,1]$.

[7]While the purpose of this example is mainly technical here, we should nevertheless note that topological graphs are not unrealistic alternative spaces. For instance, in certain types of location choice problems (where vertices of the graph are, say, cities, and the arcs of the graph represent the roads between the cities) such graphs serve as natural alternative spaces.

3.3 Discussion. As pioneered by Hendrik Houthakker and Paul Samuelson, the earlier work in revealed preference theory was conducted within restricted classes of choice problems (such as budget sets in $\mathbb{R}^n_+$). The seminal contributions of Arrow (1959) and Richter (1966, 1971) have shifted the focus of decision theorists to studying the consequences of rational decision-making in richer settings. Especially the recent literature on boundedly rational choice theory (that aim at studying choice behavior that may exhibit phenomena such as the attraction effect, shortlisting, indecisiveness, etc.) has adopted a common framework in which the "collection of all choice problems" is a set of subsets of a universal set X that contains $\mathcal{F}(X)$. In fact, a good deal of papers in this literature take X to be a finite set, and work with choice *functions* on $\mathcal{F}(X)$.[8] On the positive side, this simplifies the involved axiomatic analysis substantially. On the negative side, however, it hides the fact that "single-valuedness" of a choice correspondence is itself a behavioral axiom, and one that may have strong implications. Indeed, it is often difficult to extend such studies to the context of multi-valued choice correspondences; it is usually not even clear how to modify the associated behavioral postulates in that context.

Having said this, we should note that the "ideal" framework of revealed preference analysis depends on the sort of behavioral phenomena that one is after. If what is under investigation is procedural ways of making choices, it is quite natural to adopt a model in which one's choice behavior is modeled by means of a choice *function*. (And, to be fair, most of the papers we have mentioned above, but not Kalai, Rubinstein and Spiegler (2002) and Xu and Zhou (2007), are of this form.) In such cases, the fact that the model may end up yielding discontinuous choice behavior is of little concern, for, after all, continuity is a technical property and "procedures" are rarely continuous. If, on the other hand, one is after providing a foundational, say, normative, theory of choice in which continuity of the choice behavior is deemed desirable, then the results above warns us against using choice functions as primitives of analysis.

In passing, we note that one can always escape an impossibility result such as Theorem 3.1 by restricting the domain of a choice function suitably. For instance, in the context of the standard budget problems (in $\mathbb{R}^n_+$), maximization of a continuous and strictly quasi-concave utility function over a budget set always yields a unique outcome, and therefore, by the Berge Maximum Theorem, a choice function rationalized by such a utility function is continuous. The same is true for convex n-person Nash bargaining problems. If, observationally, the researcher is only privy to such a restricted domain of choice problems, then it is possible to work with continuous choice functions. In view of the results above, one should, however, keep in mind that extending the domain may well rule out such choice functions. This happens, for instance, if we wish to include nonlinear budget problems (that arise, say, due to quantity taxes) in the context of the budget problems, or non-convex bargaining problems (that arise, say, because bargainers may have non-expected utility preferences) in the context of Nash bargaining problems (see Remark 3.1).

3.4 An Application to the Theory of Incomplete Preferences. Let X be a metric space. By a *preference relation* on X we mean a reflexive and transitive binary relation on X, and by a *strict preference relation* on X we mean an asymmetric and transitive binary relation on X. The *strict part* of a preference relation $\succsim$ on X is the binary relation $\succ$ on X defined by $x \succ y$ iff $x \succsim y$ but not $y \succsim x$. (Obviously, $\succ$ is a strict preference relation on X.)

A preference relation is said to be *continuous* if it is a closed subset of $X \times X$, while a strict preference relation is called *continuous* if it is an open subset of $X \times X$. These are consistent under the completeness hypothesis, in the sense that a complete preference relation $\succsim$ on X is continuous iff $\succ$ is a continuous strict preference relation on X. For incomplete preferences, however, the situation is different. Indeed, a famous result of preference theory, due to Schmeidler (1971), says that, when X is connected, the strict part $\succ$ of a continuous preference relation $\succsim$ is (nonempty and) continuous only if $\succsim$ is complete. As an immediate application of Theorem 3.1, we observe here that for preference relations that lack nontrivial indifference parts, strengthening the connectedness hypothesis in this theorem to path-connectedness (or to connectedness and compactness) strengthens the conclusion of Schmeidler's

[8]There are many such examples among which are Apesteguia and Ballester (2012), Green and Hojman (2012), Kalai, Rubinstein and Spiegler (2002), Manzini and Mariotti (2007), Cherepanov, Feddersen and Sandroni (2012), Masatlioglu, Nakajima and Ozbay (2012), and Xu and Zhou (2007).

theorem substantially.

Proposition 3.3. Let X be a path-connected (or connected and compact) metric space, and $\succsim$ an antisymmetric preference relation on X such that $\succ \neq \emptyset$. If both $\succsim$ and $\succ$ are continuous, then $\succsim$ is complete and X is homeomorphic to a real interval.

Indeed, by Schmeidler's theorem, $\succsim$ is a continuous and complete partial order on X. Then, $c : \mathcal{F}(X) \to X$, defined by setting $c(S)$ as the maximum element of S with respect to $\succsim$, is a choice function on $\mathcal{F}(X)$. As it is readily verified that the continuity of $\succsim$ implies that of c, Proposition 3.3 follows upon applying Theorem 3.1.

3.5 Supplementary Remarks.

Remark 3.2. Theorem 3.1 is not a density type result that builds on the fact that we allow for *all* finite subsets of X in the domain of c. That is, $\mathcal{F}(X)$ can be replaced with $\mathcal{F}_k(X)$ in the statement of Theorem 3.1 so long as $k \geq 2$. (In fact, the proof we give below for Theorem 3.1 uses only $\mathcal{F}_2(X)$.[9]) Insofar as the problems that an individual may face at any one time are concerned, the setting in which Theorem 3.1 is valid is as finitistic as possible. (At the other extreme, it is plain that $\mathcal{F}(X)$ can be replaced with any of its supersets within $\mathbf{k}(X)$.)

Remark 3.3. While connected but not path-connected metric spaces are rather esoteric spaces (such as the so-called *topologist's sine curve*), one may still wonder if Theorem 3.1 is tight with respect to its topological assumptions. In particular, it is natural to ask if path-connectedness can be relaxed to connectedness, or if compactness can be relaxed to separability, in the context of this result. Surprisingly, the answers to these questions are both negative. We prove this in Section 4.5 by providing an example of a continuous choice function on $\mathcal{F}(X)$ where X is a connected and separable metric space of arbitrarily large topological dimension.

Remark 3.4. Theorem 3.1 provides a novel outlook to the recent work of Magyarkuti (2010) who shows that, given a connected metric space X and a choice function c on $\mathcal{F}(X)$, the continuity of c implies the asymmetry of the binary relation $\succ$, defined on $X \times X$ by $x \succ y$ iff there is an $S \in \mathcal{F}(X)$ with $x = c(S)$ and $y \in S \backslash \{x\}$. This, in turn, guarantees that c, when continuous, satisfies the Weak Axiom of Revealed Preference (WARP). At first glance this appears to be a surprising result, for it deduces a behavioral property from a topological condition such as continuity. But, in fact, "single-valuedness" of c (which is a behavioral property) is largely responsible for this fact. In particular, Theorem 3.1 shows that for path-connected (or compact and connected) X, Magyarkuti's theorem is void of content, unless X is homeomorphic to an interval in $\mathbb{R}$.

Remark 3.5. Given a metric space X, the issue of whether or not a continuous choice function on $\mathcal{F}(X)$ satisfies WARP is only indirectly related to Theorem 3.1. Such a choice function must satisfy this property when X is connected, but this fact (which is not used in the proof of Theorem 3.1.b) is not the driving force behind Theorem 3.1. After all, there are continuous choice functions on $\mathcal{F}(X)$ which satisfy WARP, even though X can be of arbitrarily large dimension (Section 4.5). Furthermore, a continuous choice function on $\mathcal{F}(X)$ need not satisfy WARP when X is compact, but Theorem 3.1.a shows that the existence of such a function nevertheless forces the (topological) embedding of X in $\mathbb{R}$.

4 ANALYSIS

[9] This may be important for certain applications. For instance, consider those correspondences that are rationalized as top cycles of a complete and asymmetric binary relation. (In the case where X is finite, such choice correspondences are characterized by Ehlers and Sprumont (2007).) More precisely, let X be any nonempty set, R a complete and asymmetric binary relation on X, and define $c : \mathcal{F}(X) \to 2^X$ by $c(S)$ being the set of all maximal elements in S with respect to the transitive closure of $R \cap (S \times S)$. This choice correspondence is single-valued on $\mathcal{F}_2(X)$, so it is subject to Theorem 3.1.

This section is devoted to proving Theorem 3.1 and establishing a few useful side results. Before we proceed with the formal argument, however, we outline the basic thread of the proof.

Let X be a metric space, and c a choice function on $\mathcal{F}_2(X)$. Let $\succcurlyeq$ be the binary relation on X that relates an alternative to another iff that alternative is chosen over the other in the associated pairwise comparison. It is easy to see that $\succcurlyeq$ is complete and antisymmetric, and if c is continuous, it is closed in $X \times X$, but short of the Weak Axiom of Revealed Preference at hand, we cannot conclude that $\succcurlyeq$ is transitive. However, connectedness of X and continuity of c jointly ensure the transitivity of $\succcurlyeq$, an observation due to Eilenberg (1941). So, with connectedness of X and continuity of c, we find that $\succcurlyeq$ is a continuous linear order on X. (The existence of such a linear order when X is compact, is a result we owe to van Mill and Wattel (1981).) The hardest part of the proof is to show that the existence of a continuous linear order on a path-connected metric space ensures the separability of that space. (When X is compact, this is trivially true.) Once this is done we may use a standard utility representation theorem to find a continuous injection into $\mathbb{R}$. The final step of the proof is then to show that every continuous injection from a path connected metric space into $\mathbb{R}$ is in fact a homeomorphism onto its range.

Nomenclature. For any nonempty set X, $x \in X$, and a binary relation R on X, we define

$$x^{\uparrow,R} := \{\omega \in X : \omega \; R \; x\} \quad \text{and} \quad x^{\downarrow,R} := \{\omega \in X : x \; R \; \omega\}.$$

In addition, for any x and y in X, we denote the R-interval $y^{\uparrow,R} \cap x^{\downarrow,R}$ by $[y,x]^R$.

By a *poset* (partially ordered set) we mean a pair $(X, \succcurlyeq)$ where X is a nonempty set and $\succcurlyeq$ a partial order on X. The asymmetric part of $\succcurlyeq$ is denoted as $\succ$. As a notational convention, we write $(y,x]^{\succcurlyeq}$ for $[y,x]^{\succcurlyeq} \backslash \{y\}$, and $[y,x)^{\succcurlyeq}$ for $[y,x]^{\succcurlyeq} \backslash \{x\}$. A *loset* (linearly ordered set) is a poset whose partial order is complete.

4.1 A Separability Theorem for Metric Spaces. To streamline the proof of Theorem 3.1, we isolate here the key step of the argument in the form of a general lemma. The proof of this result is a bit involved, and it is relegated to Section 4.5.

Lemma 4.1. If there exists a continuous linear order on a path-connected metric space, then that space is separable.[10]

4.2 Proof of Theorem 3.1.a. Let X be a compact metric space and assume that there is a continuous choice function on $\mathcal{F}(X)$. Then, in the terminology of van Mill and Wattel (1981), X has a weak selection, namely, $c|_{\mathcal{F}_2(X)}$, and hence, by Theorem 1.1 of that paper, there is a continuous linear order $\succcurlyeq$ on X. As every compact metric space is separable, we may apply the Debreu (Utility) Representation Theorem to obtain a continuous utility function $u : X \to \mathbb{R}$ that represents $\succcurlyeq$. As $\succcurlyeq$ is antisymmetric, u is injective, that is, u is a continuous bijection from X onto its range. As X is compact, u must be a homeomorphism, and we are done.

4.3 Proof of Theorem 3.1.b. Let X be a connected metric space. Assume that there is a homeomorphism f from X onto an interval in $\mathbb{R}$. Then, $c : \mathcal{F}(X) \to X$, defined by $c(S) := f^{-1}(\max f(S))$, is a continuous choice function on $\mathcal{F}(X)$. We shall use the following result for establishing the converse claim.

Lemma 4.2.[11] Let X be a connected metric space and c a continuous choice function on $\mathcal{F}_2(X)$. Then, there is a continuous linear order on X.

[10] The fact that we work with metric spaces is essential here. This result is not true for path-connected Hausdorff topological spaces. For instance, the so-called *long line* is a (non-metrizable) non-separable path-connected Hausdorff space relative to the order topology induced by its linear order, and the linear order of this space is continuous.

[11] This result is not novel. It is contained in the conjunction of Theorem 2.1 of Eilenberg (1941) and Lemma 7.2 of Michael (1951), but we provide a short direct proof.

Proof. Define the binary relation $\succcurlyeq$ on X by $x \succcurlyeq y$ iff $x = c\{x, y\}$. It is plain that $\succcurlyeq$ is reflexive, antisymmetric and complete. Moreover, it is readily checked that the continuity of c implies that $\succcurlyeq$ is a closed subset of $X \times X$. In particular, $x^{\uparrow,\succcurlyeq}$ and $x^{\downarrow,\succcurlyeq}$ are closed in X for every $x \in X$. By duality (that is, because $(x, y) \mapsto (y, x)$ is a homeomorphism from $X \times X$ onto itself), $\preccurlyeq := \{(y, x) : x \succcurlyeq y\}$ is closed in $X \times X$ as well. As the completeness and antisymmetry of $\succcurlyeq$ ensure that $\succ$ equals $(X \times X) \setminus \preccurlyeq$, therefore, $\succ$ is open in $X \times X$. In particular, $x^{\uparrow,\succ}$ and $x^{\downarrow,\succ}$ are open in X for every $x \in X$. We now claim that $\succcurlyeq$ is transitive. Indeed, if x, y and z are elements of X such that $x \succ y$, $y \succ z$ and $z \succ x$, then $[y, z]^{\succ}$ is an open subset of X which is nonempty (because it contains x). But $[y, z]^{\succ} = [y, z]^{\succcurlyeq}$ (because $\succcurlyeq$ is antisymmetric), so $[y, z]^{\succ}$ is in fact a nonempty proper clopen subset of X, contradicting the connectedness of X. Conclusion: $\succcurlyeq$ is a continuous linear order on X. $\square$

Now, assume that X is path-connected, and that there exists a continuous choice function on $\mathcal{F}(X)$. Then, by Lemma 4.2 there is a continuous linear order $\succcurlyeq$ on X, so Lemma 4.1 implies that X is a separable metric space. We may thus apply Theorem 6.1 of Eilenberg (1941) to find a continuous injection $u : X \to \mathbb{R}$ such that $x \succcurlyeq y$ iff $u(x) \geq u(y)$ for every $x, y \in X$. The proof is then concluded by applying the following observation.

Lemma 4.3. Let X be a path-connected metric space such that there exists a continuous injection u from X into $\mathbb{R}$. Then, u is a homeomorphism from X onto $u(X)$.

Proof. If X is a singleton there is nothing to prove, so we assume $|X| > 1$. As X is connected, $u(X)$ is a connected subset of $\mathbb{R}$, so $u(X)$ is then a nondegenerate interval in $\mathbb{R}$. Let t be any number in this interval. If $t = \min u(X)$, we set $a := t$ and pick any $b \in u(X)$ such that $[a, b] \subseteq u(X)$, and if $t = \max u(X)$, we set $b := t$ and pick any $a \in u(X)$ such that $[a, b] \subseteq u(X)$. If, on the other hand, t belongs to the interior of $u(X)$, we pick any $a, b \in u(X)$ such that $a < t < b$ and $[a, b] \subseteq u(X)$. Now put $x := u^{-1}(a)$ and $y := u^{-1}(b)$. As X is path-connected, there exists a path joining x to y, that is, there is a continuous map $f : [0, 1] \to X$ such that $f(0) = x$ and $f(1) = y$. Clearly, $f[0, 1]$ is a compact and connected subset of X, so $u(f[0, 1])$ must be a compact interval in $\mathbb{R}$. As x and y belong to $f[0, 1]$, this interval contains $[u(x), u(y)]$, that is, $[a, b] \subseteq u(f[0, 1])$. But, as $f[0, 1]$ is compact, and every continuous bijection from a compact metric space onto another is a homeomorphism, $u|_{f[0,1]}$ is a homeomorphism from $f[0, 1]$ onto $u(f[0, 1])$. In particular, u^{-1} is continuous on the interval $[a, b]$, and hence, u^{-1} is continuous at t. As t was arbitrarily chosen in $u(X)$, this proves that u^{-1} is a continuous map from $u(X)$ onto X. $\square$

4.5 Proof of Lemma 4.1.

Nomenclature. For any poset $(X, \succcurlyeq)$ and for any subsets A and B of X, we write $A \succcurlyeq B$ to mean $a \succcurlyeq b$ for every $(a, b) \in A \times B$. For any subset Y of X with a $\succcurlyeq$-minimum x_0, we denote the $\succcurlyeq$-supremum of a subset S of Y *in* Y by $\bigvee_Y S$ if it exists, and set $\bigvee_Y \emptyset := x_0$ by convention. We also recall that $(X, \succcurlyeq)$ is *Dedekind complete* if $\bigvee_X Y$ exists for every nonempty $Y \subseteq X$ that is $\succcurlyeq$-bounded from above. Besides, a set Y in X is said to be $\succcurlyeq$*-cofinal* if for every $y \in X$ there is an $x \in Y$ with $x \succcurlyeq y$. It is referred to as a $\succcurlyeq$*-chain* if $\succcurlyeq \cap (Y \times Y)$ is a linear order on Y.

Finally, for any posets $(X, \succcurlyeq_X)$ and $(Y, \succcurlyeq_Y)$ in which X and Y are topological spaces, we refer to a map $f : X \to Y$ as an *order-isomorphism* if f is a bijection such that $x \succcurlyeq_X y$ iff $f(x) \succcurlyeq_Y f(y)$ for every $x, y \in X$, and say that f is an *order-homeomorphism* if f is both a homeomorphism and an order-isomorphism. If there is such an f, we say that $(X, \succcurlyeq_X)$ and $(Y, \succcurlyeq_Y)$ are *order-homeomorphic.*

We now proceed with the proof of Lemma 4.1. Let X be a path-connected metric space with $|X| > 1$, and assume that there is a continuous linear order $\succcurlyeq$ on X. In what follows, we simplify our notation by writing $x^{\downarrow}$ for $x^{\downarrow,\succcurlyeq}$ and $x^{\uparrow}$ for $x^{\uparrow,\succcurlyeq}$ for any $y \in X$. We begin with a preliminary observation.

Claim 1. For any $x, y \in X$ with $x \succ y$, the subspace $[y, x]^{\succcurlyeq}$ of X is order-homeomorphic to $[0, 1]$.

Proof of Claim 1. Pick a path g joining y to x, and put $Y := g[0, 1]$. Clearly, Y is compact and connected metric space, and $\succcurlyeq \cap (Y \times Y)$ is a continuous linear order on Y, so by Theorem 6.1 of Eilenberg (1941), there exists a continuous injection u from Y into $\mathbb{R}$ such that $z \succcurlyeq w$ iff $u(z) \geq u(w)$

for every $z, w \in Y$. As Y is compact, u must be an open map, so u is in fact an order-homeomorphism from Y onto $u(Y)$. But, for any z in $[y,x]^{\succcurlyeq}$, the sets $z^{\uparrow} \cap Y$ and $z^{\downarrow} \cap Y$ are nonempty closed proper subsets of Y, so connectedness of Y entails that these sets have a nonempty intersection, and hence, as $\succcurlyeq$ is antisymmetric, we find $z \in Y$. This shows that $[y,x]^{\succcurlyeq} \subseteq Y$. We may then define $f := u|_{[y,x]^{\succcurlyeq}}$, and note that this is an order-homeomorphism from $[y,x]^{\succcurlyeq}$ onto $u([y,x]^{\succcurlyeq})$. But, as u is an order-isomorphism, we have $u([y,x]^{\succcurlyeq}) = [u(y), u(x)]$. Conclusion: $[y,x]^{\succcurlyeq}$ is order-homeomorphic to a compact interval. $\|$

Now, if X is $\succcurlyeq$-bounded, then $X = [y,x]^{\succcurlyeq}$ for some $x, y \in X$, and hence the separability of X follows readily from Claim 1. Let us then assume that X is not $\succcurlyeq$-bounded, but it has a $\succcurlyeq$-minimum element, say, x_0.

Claim 2. For any $x \in X \backslash \{x_0\}$, the subspace $x^{\downarrow}$ of X is order-homeomorphic to $[0,1]$. Furthermore, $(X, \succcurlyeq)$ is Dedekind complete.

Proof of Claim 2. The first assertion is immediate from Claim 1. To see the second one, notice that if S is a nonempty subset of X that is $\succcurlyeq$-bounded from above, we have $\bigvee_X S = \bigvee_{x^{\downarrow}} S$ for any (fixed) $x \in X$ with $x \succcurlyeq S$. But $\bigvee_{x^{\downarrow}} S$ exists, for $x^{\downarrow}$ is a complete lattice, being order-isomorphic to $[0,1]$. $\|$

Claim 3. There exists a $\succcurlyeq$-cofinal set S in X such that $\succcurlyeq \cap (S \times S)$ is a well-order on S and

$$x = \bigvee\nolimits_X (x^{\downarrow,\succ} \cap S) \qquad \text{for every } x \in S \text{ with no } \succcurlyeq\text{-predecessor in } S. \tag{1}$$

Proof of Claim 3. (Throughout this proof, we write ${}^{\downarrow}x$ for $x^{\downarrow,\succ}$ for simplicity.) Define the binary relation $\trianglerighteq$ on 2^X by

$$A \trianglerighteq B \quad \text{ iff } \quad A \supseteq B \text{ and } A \backslash B \succcurlyeq B.$$

Clearly, $\trianglerighteq$ is reflexive and antisymmetric. If $A \trianglerighteq B$ and $B \trianglerighteq C$, then, obviously, $A \supseteq C$. Besides, $A \backslash B \succcurlyeq C$ (because $A \backslash B \succcurlyeq B$ and $B \supseteq C$), while $B \backslash C \succcurlyeq C$. As $A \backslash C = (A \backslash B) \cup (B \backslash C)$, therefore, $A \backslash C \succcurlyeq C$, that is, $A \trianglerighteq C$. Conclusion: $\trianglerighteq$ is transitive, and hence, $(2^X, \trianglerighteq)$ is a poset.

Now, we define

$$\mathcal{S} := \{S \in 2^X : \succcurlyeq \cap (S \times S) \text{ is a well-order and (1) holds}\}.$$

We will prove that $(\mathcal{S}, \trianglerighteq)$ is an inductive poset – that is, every chain in this poset has an upper bound – by showing that $\bigcup \mathcal{A} \in \mathcal{S}$ and $\bigcup \mathcal{A} \trianglerighteq \mathcal{A}$ for every $\trianglerighteq$-chain $\mathcal{A}$ in $\mathcal{S}$.

Put $T := \bigcup \mathcal{A}$. First, let us verify that $\succcurlyeq \cap (T \times T)$ is a well-order on T. Let A be a nonempty subset of T. Then $A \cap B \neq \emptyset$ for some $B \in \mathcal{A}$. As $\succcurlyeq \cap (B \times B)$ is a well-order on B, there is a $\succcurlyeq$-minimum of $A \cap B$, say, x. To derive a contradiction, suppose $x \succ y$ for some $y \in A$. As $A \subseteq T$, then, $y \in T$, so $y \in C$ for some $C \in \mathcal{A}$. As $\mathcal{A}$ is a $\trianglerighteq$-chain, either $B \trianglerighteq C$ or $C \trianglerighteq B$. In the former case, $B \supseteq C$, so $y \succcurlyeq \min(A \cap C, \succcurlyeq) \succcurlyeq \min(A \cap B, \succcurlyeq) = x$, a contradiction. In the latter case, $C \backslash B \succcurlyeq B$, so $C \backslash B \succcurlyeq A \cap B \succcurlyeq x$, which means that y does not belong to $C \backslash B$. Then, $y \in B$, and hence $y \in A \cap B$, which implies $y \succcurlyeq x$. a contradiction. Thus, $x \succ y$ for no $y \in A$, that is, x is a $\succcurlyeq$-minimum of A. Conclusion: $\succcurlyeq \cap (T \times T)$ is a well-order. Next, take any x in T with no $\succcurlyeq$-predecessor in T. We wish to prove that $x = \bigvee_X ({}^{\downarrow}x \cap T)$. To this end, pick any A in $\mathcal{A}$ that contains x. We have ${}^{\downarrow}x \cap A = {}^{\downarrow}x \cap T$. ($\subseteq$ part of this equation is obvious. Conversely, if some $y \in T \backslash A$ satisfies $x \succ y$, then for any $B \in \mathcal{A}$ that contains y we have $B \trianglerighteq A$ (because $\mathcal{A}$ is a $\trianglerighteq$-chain and B is not a subset of A), and hence $y \in B \backslash A \succcurlyeq A \ni x$, a contradiction.) It follows that x has no $\succcurlyeq$-predecessor in A. For, if z is the $\succcurlyeq$-predecessor of x in A, we have $z = \max({}^{\downarrow}x \cap A, \succcurlyeq) = \max({}^{\downarrow}x \cap T, \succcurlyeq)$, which means that z is the $\succcurlyeq$-predecessor of x in T. Since A belongs to $\mathcal{S}$, therefore, $x = \bigvee_X ({}^{\downarrow}x \cap A) = \bigvee_X ({}^{\downarrow}x \cap T)$, as we sought. Conclusion: $T \in \mathcal{S}$. It remains to show that $T \trianglerighteq \mathcal{A}$. But, obviously, $T \supseteq A$ for each $A \in \mathcal{A}$. Moreover, for each A in $\mathcal{A}$, we have $T \backslash A \succcurlyeq A$. For, if $x \in T \backslash A$, then $x \in B$ for some $B \in \mathcal{A}$, and hence $B \trianglerighteq A$ (because $\mathcal{A}$ is a $\trianglerighteq$-chain and B is not a subset of A), and hence $x \succcurlyeq A$. It follows that $T \trianglerighteq \mathcal{A}$. Conclusion: $(\mathcal{S}, \trianglerighteq)$ is an inductive poset.

We now apply Zorn's Lemma to find a $\trianglerighteq$-maximal set S in $\mathcal{S}$. It remains to prove that S is $\succcurlyeq$-cofinal in X. To derive a contradiction, suppose there is a $y \in X$ such that $y \succ x$ for each $x \in S$. As $(X, \succcurlyeq)$ is Dedekind complete (Claim 2), $\bigvee_X S$ exists in X. If $\bigvee_X S$ does not belong to S, we have $S \cup \{\bigvee_X S\} \trianglerighteq S$, while if $\bigvee_X S \in S$, we have $S \cup \{y\} \trianglerighteq S$. As in either case we contradict the $\trianglerighteq$-maximality of S in $\mathcal{S}$, we may conclude that S is $\succcurlyeq$-cofinal in X. The proof of Claim 3 is complete. $\|$

Let S be as found in Claim 3. Without loss of generality, we may include x_0 in S. Next, we make $S \times [0,1)$ a loset by using the lexicographic sum of $\succcurlyeq$ and $\geq$. That is, we define the linear order $\trianglerighteq$ on $S \times [0,1)$ by $(x,s) \trianglerighteq (y,t)$ iff either $x \succ y$, or $x = y$ and $s \geq t$. In turn, we endow $S \times [0,1)$ with the order topology induced by $\trianglerighteq$.

Claim 4. X is homeomorphic to $S \times [0,1)$.

Proof of Claim 4. As S is $\succcurlyeq$-cofinal in X, and X has no $\succcurlyeq$-maximum, every element of S has a $\succcurlyeq$-successor in S which we denote by $\sigma(x)$. For each $x \in S$, we use Claim 1 to find an order-homeomorphism $f_x : [0,1] \to [x, \sigma(x)]^{\succcurlyeq}$. Next, we define $\varphi : S \times [0,1) \to X$ by $\varphi(x,t) := f_x(t)$. Our task is to show that φ is a homemorphism.

Take any distinct (x,s) and (y,t) in $S \times [0,1)$. If $x = y$, then $s \neq t$, so $f_x(s) \neq f_y(t)$ because f_x is injective. If $x \neq y$, say, $x \succ y$, then $f_x(s) \succcurlyeq f_x(0) = x \succcurlyeq \sigma(y) = f_y(1) \succ f_y(t)$, so we again find $f_x(s) \neq f_y(t)$. Conclusion: φ is injective. Next, fix an arbitrary x in X. As $\succcurlyeq$ is a well-order on S and S is $\succcurlyeq$-cofinal, the $\succcurlyeq$-minimum element of $x^{\uparrow} \cap S$, say z, exists. If $x = z$, then $\varphi(z,0) = f_x(0) = x$. If $z \succ x$, then z must have a $\succcurlyeq$-predecessor in S, for otherwise, $\bigvee_X (z^{\downarrow,\succ} \cap S) = z \succ x$, and hence $z \succ w \succ x$ for some $w \in S$, contradicting the choice of z. Let y be the $\succcurlyeq$-predecessor of z in S. Then $\sigma(y) = z$, so $\varphi(y,[0,1)) = [y,z)^{\succcurlyeq}$. But since $z = \min(x^{\uparrow} \cap S, \succcurlyeq)$ and $y \in S$, we cannot have $y \in x^{\uparrow}$, that is, $x \succ y$, which means that $x \in [y,z]^{\succ}$, so $x \in \varphi(y,(0,1))$. Conclusion: φ is bijective. Besides, it is readily checked that $(x,s) \trianglerighteq (y,t)$ iff $f_x(t) \succcurlyeq f_y(s)$ for every (x,s) and (y,t) in $S \times [0,1)$. Conclusion: φ is an order-isomorphism.

Take any (x,s) in $S \times [0,1)$. Then, as φ is an order-isomorphism, $\varphi((x,s)^{\downarrow,\triangleright}) = \varphi(x,s)^{\downarrow,\succ} = f_x(s)^{\downarrow,\succ}$, and similarly, $\varphi((x,s)^{\uparrow,\triangleright}) = f_x(s)^{\uparrow,\succ}$, so φ maps any set of the form $(x,s)^{\downarrow,\triangleright}$ or $(x,s)^{\uparrow,\triangleright}$ to open sets in X. Since the order topology on $S \times [0,1)$ is generated by a subbase that consists of such sets, we conclude: φ is an open map. It remains to prove that φ is continuous. To this end, take any net $(x_\tau, s_\tau)_{\tau \in \mathbb{D}}$ in $S \times [0,1)$ that converges to some $(x,s) \in S \times [0,1)$. Let O be an open neighborhood of $f_s(x)$. We wish to show that $\varphi(x_\tau, s_\tau) \subseteq U$, eventually for all τ. If $s > 0$, continuity of f_x ensures that there is an interval $I \subseteq (0,1)$ such that $t \in I$ and $f_x(I) \subseteq O$. Since $\{(x,t) : t \in I\}$ is an open neighborhood of (x,s), the net $(x_\tau, s_\tau)_{\tau \in \mathbb{D}}$ will remain in this neighborhood eventually, which implies that $\varphi(x_\tau, s_\tau)$ will remain within O eventually. Assume, then, $s = 0$. There are then two cases to consider. First, suppose x has a $\succcurlyeq$-predecessor, say, y, in S. In this case, $f_y(1) = f_x(0) \in O$, and by continuity of f_x and f_y, there exist numbers a and b in $(0,1)$ such that the sets $f_y(a,1)$ and $f_x[0,b)$ are contained in O. Since $[(y,a),(x,b)]^{\triangleright}$ is an open neighborhood of $(x,0)$, we must have $(x_\tau, s_\tau) \in [(y,a),(x,b)]^{\triangleright}$, and hence,

$$\varphi(x_\tau, s_\tau) \in \varphi([(y,a),(x,b)]^{\triangleright}) = [\varphi(y,a), \varphi(x,b)]^{\succ},$$

that is, $\varphi(x_\tau, s_\tau) \in f_y(a,1) \cap f_x[0,b) \subseteq O$, eventually for all τ. Finally, suppose x does not have a $\succcurlyeq$-predecessor in S. As before, pick any $b \in (0,1)$ such that $f_x[0,b) \subseteq O$. Also, let $g : [0,1] \to X$ be an order-homeomorphism that join x_0 to x. Since g is continuous and $x = f_x(0) \in O$, there is a $t^* \in (0,1)$ such that $g(t) \in U$ for each $t \in (t^*, 1]$. Then, $(g(t^*), x]^{\succcurlyeq} \subseteq O$. But $x = \bigvee_X (x^{\downarrow,\succ} \cap S)$, so there is a $y \in S$ with $x \succ y \succ g(t^*)$. Since $[(y,0),(x,b)]^{\triangleright}$ is an open neighborhood of $(x,0)$, therefore, $(x_\tau, s_\tau) \in [(y,0),(x,b)]^{\triangleright}$, and hence,

$$\varphi(x_\tau, s_\tau) \in \varphi([(y,0),(x,b))^{\triangleright}] = [\varphi(y,0), \varphi(x,b)]^{\succ},$$

that is, $\varphi(x_\tau, s_\tau) \in (g(t^*), x]^{\succ} \cap f_x[0,b) \subseteq O$, eventually for all τ. The proof of Claim 3 is complete. $\|$

It is well-known that if S is uncountable, then $S \times [0,1)$ is not metrizable. It thus follows from Claim 4 that S is countable. Consequently, $\{(x,r) : x \in S \text{ and } r \in \mathbb{Q} \cap [0,1)\}$ is a countable dense set in $S \times [0,1)$. Thus, $S \times [0,1)$, and hence X, must be separable.

We have now established Lemma 4.1 under the hypothesis that X is not $\succcurlyeq$-bounded, but it has a $\succcurlyeq$-minimum. If X is not $\succcurlyeq$-bounded, but it has a $\succcurlyeq$-maximum, we obtain the result by applying what we have just proved to the dual order $\preccurlyeq$. If, on the other hand, X has neither a $\succcurlyeq$-minimum nor a $\succcurlyeq$-maximum, then we fix an arbitrary x in X, and notice that X is the union of $x^{\uparrow,\succcurlyeq}$ and $x^{\downarrow,\succcurlyeq}$. But path-connectedness of X ensures that both $x^{\uparrow,\succcurlyeq}$ and $x^{\downarrow,\succcurlyeq}$ are path-connected metric subspaces of X, and the restrictions of $\succcurlyeq$ to these subspaces are continuous. Therefore, applying what we have established so far to these two spaces, we may conclude that they are separable. It follows that X is separable, and the proof of Lemma 4.1 is complete.

4.5 A Counter-Example. In this section we show that there exists a continuous choice function $c : \mathbf{k}(X) \to X$ on a connected and separable metric space X whose topological dimension is arbitrarily large.[12] This, in particular, shows that one cannot relax the topological requirements of Theorem 3.1 on X to "connectedness and separability."

The first step of the argument is the construction of X by transfinite induction. Fix any positive integer $n \geq 3$, put $\mathbb{I} := [0, 1]$, and let $\mathcal{C}$ denote the collection of all closed subsets C of $\mathbb{I}^n$ such that $\pi(C)$ has the cardinality of the continuum $\mathfrak{c}$, where π is the projection map $(x_1, ..., x_n) \mapsto x_1$ from $\mathbb{I}^n$ onto $\mathbb{I}$. Let $\trianglerighteq$ be a well-order on $\mathcal{C}$ such that for each $C \in \mathcal{C}$ the cardinality of $C^{\downarrow,\triangleright}$ is strictly less than $\mathfrak{c}$, and denote by C_α the element of $\mathcal{C}$ associated with the ordinal α. Now fix an arbitrary ordinal α, and suppose, for each $\beta < \alpha$, we have found a set X_β such that (1) $\pi|_{X_\beta}$ is injective; (2) $X_\beta \cap C_\gamma \neq \emptyset$ for every $\gamma \leq \beta$; (3) $\text{card}(X_\beta) < \mathfrak{c}$; and (4) $X_\gamma \subseteq X_{\gamma'}$ whenever $\gamma \leq \gamma' \leq \beta$. Now put $Y_\alpha := \bigcup\{X_\beta : \beta < \alpha\}$, and note that $\text{card}(Y_\alpha) < \mathfrak{c}$.[13] Moreover, $\pi|_{Y_\alpha}$ is injective (by (1) and (4)), and $Y_\alpha \cap C_\beta \neq \emptyset$ for every $\beta < \alpha$. We put $X_\alpha := Y_\alpha$ if $Y_\alpha \cap C_\alpha \neq \emptyset$. Otherwise, as $\text{card}(\pi(Y_\alpha)) < \mathfrak{c}$ (because $\text{card}(Y_\alpha) < \mathfrak{c}$) and $\text{card}(C_\alpha) = \mathfrak{c}$, there is a point $x(\alpha)$ in C_α such that $\pi(x(\alpha))$ does not belong to $\pi(Y_\alpha)$. We then define $X_\alpha := Y_\alpha \cup \{x(\alpha)\}$. By the Principle of Transfinite Induction, this well-defines the collection X_α for every ordinal α with $\text{card}(\alpha) < \mathfrak{c}$; we set X to be the union of these sets. Then, (1) $\pi|_X$ is injective; and (2) $X \cap C \neq \emptyset$ for every $C \in \mathcal{C}$. By adding the missed points (if necessary), we can extend X to ensure that $\pi(X) = \mathbb{I}$, so we may assume that this is the case in what follows. Then $\pi|_X$ is a continuous bijection from X onto $\mathbb{I}$, so we may define the map $c : \mathbf{k}(X) \to X$ by $c(S) := \pi^{-1}(\max(\pi(S))$. It is easily verified that c is a continuous choice function.

Suppose $\dim X \leq n - 2$. (Here dim stands for the topological (covering) dimension.) Then, by Corollary 3.3.12 of van Mill (2001), there is a countable collection $\mathcal{O}$ of open subsets of $\mathbb{I}^n$ such that $X \subseteq \bigcap\mathcal{O}$ and $\dim(\bigcap\mathcal{O}) \leq n - 2$. For each O in $\mathcal{O}$, the sets X and $\mathbb{I}^n\backslash O$ are disjoint, so $\mathbb{I}^n\backslash O$ cannot belong to $\mathcal{C}$, that is, $\text{card}(\pi(\mathbb{I}^n\backslash O)) < \mathfrak{c}$. Since $\mathbb{I}^n\backslash\bigcap\mathcal{O} = \bigcup\{\mathbb{I}^n\backslash O : O \in \mathcal{O}\}$, therefore, $\text{card}(\pi(\mathbb{I}^n\backslash\bigcap\mathcal{O})) < \mathfrak{c}$. But then there is a $t \in \mathbb{I}$ outside $\pi(\mathbb{I}^n\backslash\bigcap\mathcal{O})$, that is, $\mathbb{I}^{n-1} = \pi^{-1}\{t\} \subseteq \bigcap\mathcal{O}$, which implies $\dim(\bigcap\mathcal{O}) \geq n - 1$, a contradiction. Conclusion: $\dim X \geq n - 1$ (and hence X cannot be homeomorphic to the real line).

As a subspace of a separable metric space is separable, X is separable. It remains to show that X is connected. To derive a contradiction, suppose there are two nonempty clopen subsets U' and V' of X such that $U' \cup V' = X$. Pick any two open sets U and V in $\mathbb{I}^n$ so that $U' = U \cap X$ and $V' = V \cap X$. These sets must be disjoint (otherwise, we can find a nondegenerate closed cube $[a, b]^n$ in $U \cap V$, and as $X \cap [a, b]^n \neq \emptyset$ (by construction), this contradicts U' and V' being disjoint). Besides, $\{\text{cl}U, \text{cl}V\}$ covers $\mathbb{I}^n$, because X is dense in $\mathbb{I}^n$ and $\{U, V\}$ covers X. We set $Y := (\text{cl}U) \cap (\text{cl}V)$. Then $Y \subseteq \mathbb{I}^n\backslash(U \cup V)$, so $X \cap Y = \emptyset$. It follows that Y cannot belong to $\mathcal{C}$, that is, $\text{card}(\pi(Y)) < \mathfrak{c}$. In particular, the interior of $\pi(Y)$ is empty.

Now, define $A := \{a \in \mathbb{I} : \{a\} \times \mathbb{I}^{n-1} \subseteq \text{cl}U\}$ and $B := \{a \in \mathbb{I} : \{a\} \times \mathbb{I}^{n-1} \subseteq \text{cl}V\}$, which are closed sets in $\mathbb{I}$. Besides, as U and V are disjoint nonempty open sets in $\mathbb{I}^n$, neither A nor B equals $\mathbb{I}$. Therefore, if either of these sets is empty, we readily find that $\mathbb{I}\backslash(A \cup B)$ is a nonempty open set in $\mathbb{I}$. On the other hand, even if both A and B are nonempty, the intersection of these sets vanishes, because if there is an $x_1 \in \mathbb{I}$ such that $\{x_1\} \times \mathbb{I}^{n-1} \subseteq (\text{cl}U) \cap (\text{cl}V) = Y$, then, since $\pi(X) = \mathbb{I}$, we can find $x_{n-1}, ..., x_n$ in $\mathbb{I}$ such that $(x_1, ..., x_n) \in X$, contradicting the disjointness of X and Y. As $\mathbb{I}$ is connected, therefore, we again find that $\mathbb{I}\backslash(A\cup B)$ is a nonempty open set in $\mathbb{I}$. But this set is contained

[12] The crux of the argument for this was kindly communicated to us by Professor Jan van Mill; we are grateful for his help.

[13] For any infinite cardinal τ, the cardinality of the union of a collection of sets of cardinality τ is again τ, provided that the cardinality of that collection is τ.

in $\pi(Y)$. (For, if $a \in \mathbb{I}\backslash\pi(Y)$, then $(\{a\} \times \mathbb{I}^{n-1}) \cap$ cl$U$ and $(\{a\} \times \mathbb{I}^{n-1}) \cap$ cl$V$ are disjoint clopen sets in $\{a\} \times \mathbb{I}^{n-1}$ (because $\{$cl$U$, cl$V\}$ covers $\mathbb{I}^n$), so one of these sets must equal $\{a\} \times \mathbb{I}^{n-1}$ (because the latter set is connected), and this means either $a \in A$ or $a \in B$.) It follows that $\pi(Y)$ has nonempty interior, contradicting what we have found in the previous paragraph.

REFERENCES

Apesteguia, J. and M. Ballester, Choice by sequential procedures, *mimeo*, Universitat Pompeu Fabra, 2012.

Arrow, K., Rational choice functions and orderings, *Economica* **26** (1959), 121-127.

Benoît, J-P. and E. A. Ok, Delay aversion, *Theoretical Econ.* **2** (2012), 71-113.

Cherepanov, V., T. Feddersen and A. Sandroni, Rationalization, *Theoretical Econ.* (2012), forthcoming.

Eilenberg, S., Ordered topological spaces, *Amer. J. Math.* **63** (1941), 39-45.

Ehlers, L. and Y. Sprumont, Weakened WARP and top cycle choice rules, *J. Math. Econ.* **44** (2008), 87-94.

Green, J. and D. Hojman, Choice, rationality and welfare measurement, *mimeo*, Harvard University, 2012.

Kalai, G., A. Rubinstein, and R. Spiegler, Rationalizing Choice Functions by Multiple Rationales, *Econometrica* **70** (2002), 2481-2488.

Magyarkuti, G., Revealed preferences: A topological approach, *J. Math. Econ.* **46** (2010), 320-325.

Manzini, P. and M. Mariotti, Sequentially rationalizable choice, *Amer. Econ. Rev.* **97** (2007), 1824-1839.

Masatlioglu, Y., D. Nakajima and E. Ozbay, Revealed attention, *Amer. Econ. Rev.* (2012), forthcoming.

Michael, E., Topologies on spaces of subsets, *Tran. Amer. Math. Soc.* **71** (1951), 152-182.

van Mill, J., *The Infinite-Dimensional Topology of Function Spaces*, North-Holland, Amsterdam, 2001.

van Mill, J. and E. Wattel, Selections and orderability, *Proc. Amer. Math. Soc.* **83** (1981), 601-605.

Ok, E.A., *Real Analysis with Economic Applications*, Princeton University Press, Princeton, 2007.

Ok, E.A. and L. Zhou, Revealed group preferences on non-convex choice problems, *Econ. Theory*, **13** (1999), 671-687.

Richter, M., Revealed preference theory, *Econometrica* **34** (1966), 635-645.

Richter, M., Rational choice, in *Preferences, Utility and Demand* (ed. by J. Chipman, et al.), Harcourt Brace Jovanovich, New York, 1971.

Schmeidler, D., A condition for completeness for partial preference relation, *Econometrica* **39** (1971), 403-404.

Xu, Y. and L. Zhou, Rationalizability of choice functions by game trees, *J. Econ. Theory* **134** (2007), 548-556.