Utility Representation of an Incomplete and Nontransitive Preference Relation*

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February 1, 2015

Abstract

The objective of this paper is to provide continuous utility representation theorems analogous to Debreu's classic utility representation theorem, albeit for preference relations that may fail to be complete and/or transitive. Specifically, we show that every (continuous and) reflexive binary relation on a (compact) metric space can be represented by means of the maxmin, or dually, minmax, of a (compact) set of (compact) sets of continuous utility functions. This notion of "maxmin multi-utility representation," generalizes the recently proposed notions of "multi-utility representation" for preorders and "justifiable preferences" for complete and quasitransitive relations. As such, our main representation theorems lead to some new characterizations of these special cases as well.

JEL Classification: D11, D81.
Keywords: Nontransitive binary relations, continuous utility, Debreu's Theorem, multi-utility representation.

1 Introduction

The notion of "utility representation" is one of the most fundamental constructs of economic theory. For a given binary relation $\mathbf{R}$ on a set X, this notion corresponds to finding a real function u on X such that

$$x \;\mathbf{R}\; y \quad \text{iff} \quad u(x) \geq u(y) \tag{1}$$

for every x and y in X. We often think of $\mathbf{R}$ as a preference relation of an individual, and interpret u as a utility function that keeps record of how this

*We thank Ozgur Evren, Kazuhiro Hara and Gil Riella for insightful discussions about the content of this paper.

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agent ranks any two alternatives. Obviously, not every binary relation can be represented in this manner. In particular, such a representation is possible only if $\mathbf{R}$ is complete and transitive. The converse is not true in general, but there are a plethora of results that provide various sufficient conditions for such $\mathbf{R}$ to admit a utility representation, and when X is a topological space, to guarantee that the map u to be found is continuous. The most famous of these results is *Debreu's Utility Representation Theorem* which says that there is a continuous real map u on X such that (1) holds for each $x, y \in X$, provided that X is a suitably well-behaved space (such as a separable metric space) and $\mathbf{R}$ is complete, transitive and continuous.

However, in many contexts, one needs to consider as a primitive "preference relation" a binary relation that may be neither complete nor transitive. In particular, since the seminal contributions of Aumann (1962) and Bewley (1986), many authors have argued that completeness is not a basic trait of rationality. There is now a fairly sizable literature on rational decision making with incomplete preferences in a variety of contexts, ranging from consumption choice to decision making under risk and uncertainty. And there is even a larger literature that works with nontransitive preferences. This literature has mostly a "boundedly rational" flavor, and it studies topics such as nontransitive indifferences that arise from perception difficulties (cf. Luce (1956)), or procedural decision making by using similarity comparisons or regret considerations (cf. Rubinstein (1988) and Loomes and Sugden (1982)), or time inconsistency that arises due to relative time discounting (cf. Roelofsma and Read (2000), and Ok and Masatlioglu (2007)), among others.[1] Besides, when we consider $\mathbf{R}$ as the preference relation of a group of individuals (as in social choice theory), it becomes only natural to allow for its lack of transitivity. We may further add to this summary by noting that in *revealed preference* theory one arrives at a "preference relation" endogenously, and in many cases of interest it is impossible to guarantee either the completeness or the transitivity of this relation. The recent literature on boundedly rational choice theory provides numerous illustrations of this situation.[2]

These considerations motivate extending Debreu's Theorem in some way to the context of binary relations that are neither complete nor transitive. Obviously, this requires us to modify the "utility representation" notion used in that theorem appropriately. In the case of incomplete, but transitive,

[1] The literatures on incomplete and nontransitive preferences is simply too large to cite here comprehensively. For numerous illustrations and citations from the first of these, we refer the reader to Evren and Ok (2011), and for those from the second to Nishimura (2015).

[2] In Eliaz and Ok (2006), for instance, revealed preference relations are incomplete, and in Cherepanov, Feddersen and Sandroni (2013), they are nontransitive. On the other hand, Manzini and Mariotti (2007), Masatlioglu and Ok (2014), and Ok, Ortoleva and Riella (2015) use the revealed preference method to obtain what they call "psychological constraint relations" which need not be either complete or transitive.

binary relations, one such notion was introduced in Ok (2002), and then later developed in Evren and Ok (2011) and Bosi and Herden (2012). This notion is called "multi-utility representation," and corresponds to finding a *collection* $\mathcal{U}$ of real functions on X such that

$$x \mathbf{R} y \quad \text{iff} \quad u(x) \geq u(y) \text{ for each } u \in \mathcal{U}, \tag{2}$$

or equivalently,

$$x \mathbf{R} y \quad \text{iff} \quad \inf_{u \in \mathcal{U}} (u(x) - u(y)) \geq 0,$$

for every x and y in X. This notion allows us to think of an incomplete preference relation $\mathbf{R}$ as arising from the unanimity of several complete preference relations each of which admits a utility representation in the standard sense. There is also an analogue of Debreu's Theorem for this notion, for it is known that there is such a set $\mathcal{U}$ that contains *continuous* real maps on X such that (2) holds for each $x, y \in X$, provided that X is a suitably well-behaved space (such as a compact metric space) and $\mathbf{R}$ is reflexive, transitive and continuous.

In this paper, we propose two weaker notions of "utility representation" which are suitable for binary relations that need not be either complete or transitive. We refer to the first of these as the "maxmin multi-utility representation." This notion corresponds to finding a *collection* $\mathbb{U}$ of *collections* of real functions on X such that

$$x \mathbf{R} y \quad \text{iff} \quad \sup_{\mathcal{U} \in \mathbb{U}} \inf_{u \in \mathcal{U}} (u(x) - u(y)) \geq 0, \tag{3}$$

for every x and y in X. Clearly, this notion reduces to that of "multi-utility representation" when $\mathbb{U}$ is a singleton. It also allows us to think of the ranking of two alternatives by any given reflexive preference relation $\mathbf{R}$ as rationalized by the unanimity of a set of complete preference relations each of which admits a utility representation in the standard sense, but unlike in the case of "multi-utility representation," it permits using different sets of such preference relations for different pairs of alternatives. It is also worth noting that this representation notion is a generalization of what Lehrer and Teper (2011) call "Knightian-justifiable preferences," and if every member of $\mathbb{U}$ is a singleton, then (3) reduces, at least conceptually, to what Lehrer and Teper (2011) call "justifiable preferences," in the context of decision making under uncertainty. The main results of our paper (Theorems 1a and 2a) show that "maxmin multi-utility representation" holds in quite general circumstances: There is such a set $\mathbb{U}$ that contains sets of *continuous* real maps on a metric space X such that (3) holds for each $x, y \in X$, provided that $\mathbf{R}$ is reflexive. Moreover, when X is compact and $\mathbf{R}$ is continuous, we can make sure that each element of $\mathbb{U}$ is a compact collection of continuous functions, and further, that $\mathbb{U}$ is itself compact (relative to the Hausdorff metric).

Our second notion of "utility representation" is dual to the first one; we call it the "minmax multi-utility representation." This notion corresponds to finding a *collection* $\mathbb{V}$ of *collections* of real functions on X such that

$$x \mathbf{R} y \quad \text{iff} \quad \inf_{\mathcal{V}\in\mathbb{V}} \sup_{v\in\mathcal{V}} (v(x) - v(y)) \geq 0, \tag{4}$$

for every x and y in X. This representation notion too takes "multi-utility representation" and "justifiable preferences" as special cases, and we show in Theorem 1b that it characterizes all reflexive binary relations on a metric space. If X is compact and $\mathbf{R}$ is continuous, we can again take $\mathbb{V}$, and every element of $\mathbb{V}$, as compact in this representation (Theorem 2b).

As an application of our method of analysis, we also revisit the continuous multi-utility representation theorem of Evren and Ok (2011), and demonstrate that the collection of utility functions can be chosen as compact in that result (Theorem 3) when the underlying alternative space is compact. Finally, we prove that every complete, quasitransitive and continuous binary relation on a compact metric space is a justifiable preference (Proposition 4), but we leave the problem of fully characterizing continuous justifiable preferences as an open problem. The paper ends with some concluding comments.

2 Preliminaries

Throughout this paper, we let X stand for a metric space which is interpreted as a universal set of alternatives. As usual, $\mathbf{C}(X)$ stands for the set of all continuous real maps on X. When X is compact, we always think of $\mathbf{C}(X)$ as a metric space relative to the sup-metric, and note that $\mathbf{C}(X)$ is separable in that case.

By a *binary relation* on X, we mean a nonempty subset $\mathbf{R}$ of $X \times X$, but, as usual, we write $x \mathbf{R} y$ to mean $(x, y) \in \mathbf{R}$. We denote the asymmetric part of this relation by $\mathbf{R}^>$ (that is, $\mathbf{R}^>$ is either empty or it is a binary relation on X such that $x \mathbf{R}^> y$ iff $x \mathbf{R} y$ but not $y \mathbf{R} x$). We recall that $\mathbf{R}$ is said to be *reflexive* if $x \mathbf{R} x$ for each $x \in X$, *complete* (or *total*) if either $x \mathbf{R} y$ or $y \mathbf{R} x$ holds for each $x, y \in X$, and *transitive* if $x \mathbf{R} y$ and $y \mathbf{R} z$ imply $x \mathbf{R} z$ for each $x, y, z \in X$. If $\mathbf{R}$ is reflexive and transitive, it is said to be a *preorder* on X. We say that $\mathbf{R}$ is *quasitransitive* if $\mathbf{R}^>$ is transitive. It is plain that every preorder on X is quasitransitive, but not conversely.

Given that X is endowed with a topological structure, there are a variety of ways of defining a notion of "continuity" for a binary relation $\mathbf{R}$ on X. We will adopt the most commonly used version of these here, and say that $\mathbf{R}$ is *continuous* if it is a closed subset of $X \times X$. This is the same thing as saying that $\lim x_m \mathbf{R} \lim y_m$ for any two convergent sequences (x_m) and (y_m) in X with $x_m \mathbf{R} y_m$ for each m.

With this terminology, we can state *Debreu's Theorem* as follows: A binary relation $\mathbf{R}$ on a compact (in fact, separable) metric space X is a continuous and complete preorder on X if, and only if, there is a map $u \in \mathbf{C}(X)$

such that $x \mathbf{R} y$ iff $u(x) \geq u(y)$ for every $x, y \in X$. All of the representation results reported in the next section can be regarded as extensions of this fundamental theorem.

3 Main Results

3.1 Utility Representation Theorems

Let $\mathbf{R}$ be a binary relation on an alternative space X, which we interpret as the preference relation of an individual. Given this interpretation, we obviously wish to assume that $x \mathbf{R} x$ for every alternative x in X. Consequently, we will focus on reflexive $\mathbf{R}$ throughout the present exposition. Moreover, as we are interested in deriving continuous "utility" functions for such relations, we always maintain that X is a metric space.

We say that $\mathbf{R}$ admits a *continuous maxmin multi-utility representation* if there is a nonempty collection $\mathbb{U}$ of nonempty subsets of $\mathbf{C}(X)$ such that

$$x \mathbf{R} y \quad \text{iff} \quad [\text{there is a } \mathcal{U} \in \mathbb{U} \text{ such that } u(x) \geq u(y) \text{ for each } u \in \mathcal{U}]$$

for every x and y in X. It is easy to see that this notion generalizes many other notions of "utility representation" studied in the literature. Let us first, however, concentrate on how we may interpret this representation notion.

For purposes of interpretation, let us refer to an agent whose preference relation on X admits a continuous utility representation as "totally rational." In this jargon, then, Debreu's Theorem says that, when X is separable, an agent whose preference relation on X is a continuous and complete preorder is totally rational.

Now, consider an individual whose preference relation $\mathbf{R}$ on X admits a continuous maxmin multi-utility representation. Obviously, this individual need not be totally rational. However, her preferences can be viewed as arising from the aggregation of certain *coalitions* of totally rational individuals in a particular manner. To wit, it is as if our individual has multiple "selves," where each "self" is a totally rational individual. The agent's decision making is guided by coalitions of these "selves." More precisely, there is a set $\mathcal{S}$ of coalitions (i.e., sets) of totally rational individuals such that $\mathbf{R} = \bigcup_{S \in \mathcal{S}} \bigcap \{\succsim_{i,S} : i \in I_S\}$, where $\succsim_{i,S}$ is the preference relation of the "self i in the coalition S." (Here, $\mathcal{S}$ is the index set of the coalitions, and for each $S \in \mathcal{S}$, I_S is the index set of the members of the coalition S.) Thus, our individual ranks x over y iff there is at least one coalition of her totally rational selves that surely recommends choosing x over y in the sense that every single "self" who belongs to this coalition says x is better than y. The preference relation $\mathbf{R}$ is thus *rationalized* in the same sense that is proposed by Cherepanov, Feddersen and Sandroni (2013), albeit, rationalization takes place here through coalitions of "rationales" – what we call a "self" here is called a "rationale" in that paper – as opposed to the "rationales" themselves.[3]

It is worth noting that the "maxmin" flavor of the notion of continuous maxmin multi-utility representation is not entirely straightforward. This would be more in the clear, for instance, if the representation required instead that we find a nonempty collection $\mathbb{U}$ of nonempty subsets of $\mathbf{C}(X)$ such that

$$x \mathbf{R} y \quad \text{iff} \quad \sup_{\mathcal{U}\in\mathbb{U}} \inf_{u\in\mathcal{U}} (u(x) - u(y)) \geq 0 \tag{5}$$

for every x and y in X. This is an alternative representation notion which is clearly related to that of continuous maxmin multi-utility representation, but it is not obvious if these notions are equivalent. What is obvious is that any relation $\mathbf{R}$ that admits either a continuous maxmin multi-utility representation or a representation of the form (5) is reflexive. Our first main result in this paper shows that the converse of this is also true. It turns out that either of these notions of utility representation is equivalent to reflexivity.

Theorem 1a. [*Maxmin Representation of* $\mathbf{R}$] Let X be a metric space and $\mathbf{R}$ a binary relation on X. Then, the following are equivalent:

(i) $\mathbf{R}$ is reflexive;

(ii) $\mathbf{R}$ admits a continuous maxmin multi-utility representation;

(iii) There is a nonempty collection $\mathbb{U}$ of nonempty subsets of $\mathbf{C}(X)$ such that (5) holds for every x and y in X.

Thus, the interpretation we gave for the notion of continuous maxmin multi-utility representation above is tenable for *any* reflexive binary relation $\mathbf{R}$ on X. Moreover, this is the same thing as the "maxmin" representation of the form (5). This is somewhat surprising as both of these representation notions work with continuous utility functions, while no continuity assumption is imposed on $\mathbf{R}$. This fact may help working with reflexive preference relations which need not be continuous, or complete, or transitive (just like working with a continuous utility function is often easier than working with a continuous preorder). Conversely, Theorem 1a specifies a general method of defining an arbitrary reflexive relation (on any metric space). Using a set of sets of continuous real functions as in either part (ii) or (iii) of Theorem 1a, apparently, exhausts all such binary relations.[4]

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[3]Put more accurately, if we agree to refer to a continuous and complete preorder on X as a "rationale," and understand from the statement $x \mathbf{R} y$ that the agent chooses x from the feasible set $\{x, y\}$, then the representation notion introduced in Theorem 1a becomes what Cherepanov, Feddersen and Sandroni (2013) call *rationalization* of a choice correspondence (on pairwise choice problems).

[4]We should note that this representation notion relates closely to what Lehrer and Teper (2011) call "Knightian-justifiable" preferences. In particular, let A and S be nonempty finite sets, and $X = \triangle(A)^S$, where $\triangle(A)$ is the set of all probability vectors on A. (We interpret any one member of X in this case as an Anscombe-Aumann act.) When any one member of $\bigcup \mathbb{U}$ is an affine map of the form $f \mapsto \sum_{s\in S} \mu(s)U(f(s))$, where U is a

One may wonder at this point if the "maxmin" approach plays a singular role in the representation of reflexive binary relations. In particular, a natural question is if we can provide a dual representation for $\mathbf{R}$ which is instead of the "minmax" form. It turns out that we can do this at this level of generality, as our next result demonstrates.

Theorem 1b. [*Minmax Representation of* $\mathbf{R}$] Let X be a metric space and $\mathbf{R}$ a binary relation on X. Then, the following are equivalent:

(i) $\mathbf{R}$ is reflexive;

(ii) There is a nonempty collection $\mathbb{V}$ of nonempty subsets of $\mathbf{C}(X)$ such that

$$x \mathbf{R} y \quad \text{iff} \quad [\text{for every } \mathcal{V} \in \mathbb{V} \text{ there is a } v \in \mathcal{V} \text{ such that } v(x) \geq v(y)]$$

for every x and y in X.[5]

(iii) There is a nonempty collection $\mathbb{V}$ of nonempty subsets of $\mathbf{C}(X)$ such that

$$x \mathbf{R} y \quad \text{iff} \quad \inf_{\mathcal{V} \in \mathbb{V}} \sup_{v \in \mathcal{V}} (v(x) - v(y)) \geq 0$$

for every x and y in X.

Adopting the jargon and notation introduced above, the equivalence of (i) and (ii) in Theorem 1b says that if $\mathbf{R}$ is reflexive, then $\mathbf{R} = \bigcap_{T \in \mathcal{T}} \bigcup \{\trianglerighteq_{i,T} : i \in J_T\}$ where $\mathcal{T}$ and J_T (for each $T \in \mathcal{T}$) are index sets, and each $\trianglerighteq_{i,T}$ is a complete preorder on X that admits a continuous utility representation. Thus, $x \mathbf{R} y$ iff in any coalition of her totally rational "selves," there is at least one "self" that says x is better than y, that is, no coalition of her totally rational "selves" may block the alternative x in favor of y. (The equivalence of (i) and (iii), on the other hand, demonstrate in exactly what way we can think of *any* reflexive binary relation as a "minmax" of multiple continuous utility functions.) This "coalitional rationalization" notion is applicable whenever the one we gave for Theorem 1a is available; deciding on which of these two notions to adopt in an application is a matter of convenience.

3.2 Continuous Utility Representation Theorems

3.2.1 Maxmin Multi-Utility Representation

The structure of the "maxmin" representation provided in part (iii) of Theorem 1a would be more amenable to analysis if we could replace the sup and inf operators in (5) with the max and min operators, respectively. It

fixed affine real map on $\triangle(A)$ and μ is a probability distribution over 2^S, and if we can replace the sup and inf with the max and min operators, respectively, the representation (5) becomes identical to Knightian-justifiability.

[5] In keeping with the terminology introduced above, we say that $\mathbf{R}$ admits a *continuous minmax multi-utility representation* whenever the statement (ii) of Theorem 1b is valid.

turns out that assuming merely that $\mathbf{R}$ is continuous and that X is compact is enough to ensure that we can do this. In fact, this is a consequence of a more general fact. When $\mathbf{R}$ is continuous, which is often viewed in decision theory as a technical, but duly reasonable, hypothesis, we can not only choose $\mathbb{U}$ to consist of only compact subsets of $\mathbf{C}(X)$, but we can choose $\mathbb{U}$ itself as a compact collection (relative to the Hausdorff metric).[6]

Theorem 2a. Let X be a compact metric space and $\mathbf{R}$ a binary relation on X. Then, $\mathbf{R}$ is reflexive and continuous if, and only if, there is a compact collection $\mathbb{U}$ of nonempty compact subsets of $\mathbf{C}(X)$ such that

$$x \mathbf{R} y \quad \text{iff} \quad \max_{\mathcal{U}\in\mathbb{U}} \min_{u\in\mathcal{U}} (u(x) - u(y)) \geq 0 \tag{6}$$

for every x and y in X.

As reflexive and continuous binary relations are routinely encountered in practice, this result seems to have a fairly wide range of applicability. For instance, suppose we need to specify such a binary relation with certain properties (say, for constructing an example of some sort). Since Theorem 2a provides one with an exhaustive way of "prescribing" a reflexive and continuous binary relation, it translates this problem into choosing a particular compact collection $\mathbb{U}$ of sets of continuous real functions, which is likely to be an easier task.

3.2.2 Minmax Multi-Utility Representation

In view of the dual nature of Theorems 1a and 1b, it is natural to expect a result dual to Theorem 2a for reflexive and continuous binary relations. The following result shows that this is indeed the case.

Theorem 2b. Let X be a compact metric space and $\mathbf{R}$ a binary relation on X. Then, $\mathbf{R}$ is reflexive and continuous if, and only if, there is a nonempty compact collection $\mathbb{V}$ of nonempty compact subsets of $\mathbf{C}(X)$ such that

$$x \mathbf{R} y \quad \text{iff} \quad \min_{\mathcal{V}\in\mathbb{V}} \max_{v\in\mathcal{V}} (v(x) - v(y)) \geq 0 \tag{7}$$

for every x and y in X.

[6] Let Z be a metric space, and denote the set of all nonempty compact subsets of Z by $\mathbf{k}(Z)$. We denote the Hausdorff metric, and convergence with respect to this metric, on $\mathbf{k}(Z)$ by d^{H} and $\to^{\mathrm{H}}$, respectively. We recall that d^{H} is a uniform metric in the sense that $d^{\mathrm{H}}(S,T) = \sup\{|d(z,S) - d(z,T)| : z \in Z\}$ for any nonempty compact subsets S and T of Z, where d is the metric of the underlying space Z. (In the present discussion, Z is $\mathbf{C}(X)$, and d is the sup-metric.)

3.2.3 Multi-Utility Representation

The special case of Theorem 2a in which the binary relation at hand is not only reflexive, but it is also transitive, has recently been studied by Evren and Ok (2011). In particular, a special case of Theorem 1 of that paper says that every continuous preorder $\mathbf{R}$ on a compact metric space admits a *continuous multi-utility representation* in the sense that there is a nonempty set $\mathcal{U}$ of continuous (utility) functions on X such that

$$x \mathbf{R} y \quad \text{iff} \quad [u(x) \geq u(y) \text{ for each } u \in \mathcal{U}]$$

for every x and y in X. This suggests that, when $\mathbf{R}$ is transitive, we may be able to take $\mathbb{U}$ as a singleton in the statement of Theorem 2a above. It turns out that this is exactly the case.

Theorem 3. Let X be a compact metric space and $\mathbf{R}$ a binary relation on X. Then, $\mathbf{R}$ is a continuous preorder if, and only if, there is a compact subset $\mathcal{U}$ of $\mathbf{C}(X)$ such that

$$x \mathbf{R} y \quad \text{iff} \quad \min_{u \in \mathcal{U}} (u(x) - u(y)) \geq 0$$

for every x and y in X.

Theorem 3 is not covered by the results of Evren and Ok (2011). Indeed, this result sharpens the main representation theorem of that paper by showing that the multi-utility set $\mathcal{U}$ can be chosen as compact for the representation of a continuous preorder that is defined on a compact metric space.

3.3 Justifiable Preferences

Adopting the jargon proposed by Lehrer and Teper (2011) in the context of decision-making under uncertainty, we say that a binary relation $\mathbf{R}$ on a metric space X is a *(continuous) justifiable preference* on X if there is a collection $\mathcal{V}$ of (continuous) real maps on X such that

$$x \mathbf{R} y \quad \text{iff} \quad [v(x) \geq v(y) \text{ for some } v \in \mathcal{V}] \tag{8}$$

for every x and y in X. Obviously, every justifiable preference is complete.

Given the standing of Theorem 3 relative to Theorem 2a, it is tempting to think that imposing the completeness of $\mathbf{R}$ in Theorem 2b would entail that $\mathbf{R}$ is a continuous justifiable preference. This is, however, not correct. Indeed, if $\mathcal{V}$ represents $\mathbf{R}$ as in (8), then $x \mathbf{R}^{>} y$ holds iff $v(x) > v(y)$ for each $v \in \mathcal{V}$. Thus, any justifiable preference on X is, by necessity, quasitransitive. This leads us to the following:

Conjecture. Let X be a compact metric space and $\mathbf{R}$ a continuous binary relation on X. Then, $\mathbf{R}$ is (continuous) complete and quasitransitive if, and only if, it is a (continuous) justifiable preference on X.

We are, unfortunately, quite far from determining the validity of this conjecture at present. The following proposition comes somewhat close to saying that every continuous, complete and quasitransitive binary relation on a compact (even separable) metric space X is a justifiable preference on X, but we do not know if such a binary relation is necessarily a *continuous* justifiable preference on X.

Proposition 4. Let X be a compact metric space and $\mathbf{R}$ a continuous binary relation on X.[7] Then, $\mathbf{R}$ is complete and quasitransitive if, and only if, there is a collection $\mathcal{V}$ of real functions on X such that

$$x \mathbf{R} y \quad \text{iff} \quad \max_{v \in \mathcal{V}} (v(x) - v(y)) \geq 0 \tag{9}$$

for every x and y in X. Moreover, $\mathcal{V}$ can be chosen to contain at least one upper semicontinuous function.

Obtaining characterizations of justifiable preferences, and continuous justifiable preferences, in general, and determining whether or not our conjecture above is valid in particular, remain as open problems.

4 Concluding Comments

In this paper, we have proposed two dual notions of representation for reflexive binary relations in terms of continuous (utility) functions. To the best of our knowledge, these notions are novel, and they indeed enjoy quite a nice "multi-selves" type interpretation. They also generalize the utility representation models that are commonly adopted for complete preorders, preorders, and complete and quasitransitive binary relations. Furthermore, the main results of this paper have demonstrated that both of these representations hold under extremely weak conditions.

We are not aware of any utility representation notion for preferences that are known only to be reflexive. We should, however, mention the work of Shafer (1974) at this point who has proposed to represent a *complete* and continuous binary relation $\mathbf{R}$ on a metric space X by a continuous and skew-symmetric real map κ on $X \times X$ such that $x \mathbf{R} y$ iff $\kappa(x, y) \geq 0$. This can easily be adapted to case of continuous and reflexive relations. Moreover, it is very easy to obtain the following generalization of Shafer's representation theorem:

Proposition 5. Let X be a compact metric space and $\mathbf{R}$ a binary relation on X. Then, $\mathbf{R}$ is reflexive and continuous if, and only if, there is a continuous map $\kappa : X \times X \to \mathbb{R}$ such that $\kappa|_{\triangle_X} = 0$ and

$$x \mathbf{R} y \quad \text{iff} \quad \kappa(x, y) \geq 0 \tag{10}$$

[7] It is enough to take X as separable in this result.

for every x and y in X.

Proof. The "if" part of the result is immediate, because reflexivity of $\mathbf{R}$ is ensured by the condition $\kappa|_{\triangle_X} = 0$ while the continuity of $\mathbf{R}$ is an immediate consequence of that of κ. Conversely, assume that $\mathbf{R}$ is reflexive and continuous, and define $\kappa : X \times X \to \mathbb{R}$ by $\kappa(x, y) := -d((x, y), \mathbf{R})$, where d is the product metric of $X \times X$. As $\mathbf{R}$ is closed in $X \times X$, (10) is satisfied, while d is continuous. ■

Shafer's representation notion, however, lacks the structure of the representation models we have worked with in this paper. In particular, there is no notion of "utility function" in this representation, which makes this concept quite disconnected from the notions of "utility representation," "multi-utility representation," and "justifiable preferences." Finally, we are unable to give a behavioral interpretation to it, nor does this representation concept lead to a nontrivial representation theorem.

To conclude, we should note that we have also looked at two important special cases of our models of representation in this paper. First, we revisited the representation of a continuous preorder on a compact metric space, and showed that one can guarantee the compactness of the representing set of utility functions, thereby sharpening the main theorem of Evren and Ok (in the compact case). Second, we have examined how one may characterize (continuous) justifiable preferences. Unfortunately, we leave that job quite incomplete at present – settling the open problems we have stated for such preferences in Section 3.3 seem to require different proof techniques than those we have explored here.

APPENDIX

Notation. Let X be any nonempty set and $x, y \in X$. In what follows, for any nonempty collection $\mathcal{U}$ of real maps on X, we write $\mathcal{U}(x) \geq \mathcal{U}(y)$ to mean $u(x) \geq u(y)$ for every $u \in \mathcal{U}$. Put differently, by definition,

$$\mathcal{U}(x) \geq \mathcal{U}(y) \quad \text{ iff } \quad \inf_{u \in \mathcal{U}} (u(x) - u(y)) \geq 0.$$

Proof of Theorem 1a

Suppose (i) holds. Let us put $\succsim_{xy} := \{(x, y)\} \cup \triangle_X$, and

$$\mathcal{U}_{xy} := \{u \in \mathbf{C}(X) : u(x) \geq u(y)\}.$$

Then,

$$a \succsim_{xy} b \quad \text{ iff } \quad \mathcal{U}_{xy}(a) \geq \mathcal{U}_{xy}(b) \tag{11}$$

for any a and b in X. The "only if" part of this statement is immediate from the definitions of $\succsim_{xy}$ and $\mathcal{U}_{xy}$. To see its "if" part, suppose $a \succsim_{xy} b$ is false. Pick a map $v : \{x, y, a, b\} \to \mathbb{R}$ such that $v(b) \geq v(x) \geq v(y) \geq v(a)$ with at least one of these inequalities being strict. (Since $a \succsim_{xy} b$ is false, we have $a \neq b$ and $(a, b) \neq (x, y)$, and this ensures that such a map exists.) Now extend v to a continuous real map u on X (say, by the Tietze extension Theorem), and note that $u \in \mathcal{U}_{xy}$ and $u(b) > u(a)$, which means that $\mathcal{U}_{xy}(a) \geq \mathcal{U}_{xy}(b)$ is false. We thus conclude that (11) holds for any $a, b \in X$. But it is plain that

$$\mathbf{R} = \bigcup \{\succsim_{xy} : (x, y) \in \mathbf{R}\}.$$

Therefore, setting $\mathbb{U} := \{\mathcal{U}_{xy} : (x, y) \in \mathbf{R}\}$, we obtain (ii).

Next, to prove (iii), take any a and b in X. If $a\ \mathbf{R}\ b$, part (ii) implies readily that $\sup_{\mathcal{U}\in\mathbb{U}} \inf_{u\in\mathcal{U}}(u(a) - u(b)) \geq 0$. Conversely, suppose $a\ \mathbf{R}\ b$ is false. Then, by what we have found in the previous paragraph, for any (arbitrarily fixed) $\mathcal{U} \in \mathbb{U}$ there is a $u \in \mathcal{U}$ such that $u(b) > u(a)$. But, obviously, our construction of $\mathbb{U}$ entails that $\lambda u \in \mathcal{U}$ for any $\lambda > 0$, and this implies that $\inf_{u\in\mathcal{U}}(u(a) - u(b)) = -\infty$. As $\mathcal{U}$ is arbitrary in $\mathbb{U}$ in this observation, we thus find that $\sup_{\mathcal{U}\in\mathbb{U}} \inf_{u\in\mathcal{U}}(u(a) - u(b)) = -\infty < 0$. We conclude that (iii) holds.

We have now proved that (i) implies both (ii) and (iii). As it is obvious that each of (ii) and (iii) implies (i), Theorem 1a is proved. ■

Proof of Theorem 1b

Suppose (i) holds, and use Theorem 1a to find a collection $\mathbb{U}$ of nonempty subsets of $\mathbf{C}(X)$ such that

$$x\ \mathbf{R}\ y \quad \text{iff} \quad [\text{there is a } \mathcal{U} \in \mathbb{U} \text{ such that } u(x) \geq u(y) \text{ for each } u \in \mathcal{U}] \tag{12}$$

for every x and y in X. Moreover, as the proof of Theorem 1a shows, we can choose $\mathbb{U}$ here such that $\lambda\mathcal{U} \subseteq \mathcal{U}$ for every $\lambda > 0$ and $\mathcal{U} \in \mathbb{U}$. Now, (12) ensures that, for each $(x, y) \in X^2\backslash\mathbf{R}$ and $\mathcal{U} \in \mathbb{U}$, we can pick a $u_{x,y,\mathcal{U}} \in \mathcal{U}$ such that $u_{x,y,\mathcal{U}}(y) > u_{x,y,\mathcal{U}}(x)$. In fact, due to the invariance of each $\mathcal{U} \in \mathbb{U}$ under positive scalar multiplication, we can choose each $u_{x,y,\mathcal{U}}$ here so that $u_{x,y,\mathcal{U}}(y) = u_{x,y,\mathcal{U}}(x) + 1$. Define $\mathcal{V}_{x,y} := \{u_{x,y,\mathcal{U}} : \mathcal{U} \in \mathbb{U}\}$ and $\mathbb{V} := \{\mathcal{V}_{x,y} : (x, y) \in X^2\backslash\mathbf{R}\}$. It is easily checked that

$$a\ \mathbf{R}\ b \quad \text{implies} \quad [\text{for every } \mathcal{V} \in \mathbb{V} \text{ there is a } v \in \mathcal{V} \text{ such that } v(a) \geq v(b)]$$

and

$$\text{not } a\ \mathbf{R}\ b \quad \text{implies} \quad [\text{there is a } \mathcal{V} \in \mathbb{V} \text{ such that } v(b) = v(a) + 1 \text{ for each } v \in \mathcal{V}]$$

for any a and b in X. It is then easily deduced from these findings that both (ii) and (iii) of Theorem 1b hold. As it is obvious that either (ii) or (iii) implies (i), Theorem 1b is proved. ■

Some Observations on Lipschitz Maps

For any metric space X, we make $\mathbf{C}(X)$ into a metric space by means of the *uniform metric* d_∞ on $\mathbf{C}(X)$ which is defined by

$$d_\infty(f, g) := \sup_{x\in X} \min\{1, |f(x) - g(x)|\}.$$

Let d stand for the metric of X. For any real map f on X, we define the *Lipschitz constant* of f as

$$\text{Lip}(f) := \sup \frac{|f(x) - f(x)|}{d(x, y)}$$

where the supremum is taken over all (x, y) in $X \times X$ with $x \neq y$. For any nonnegative real number k, we say that f is *k-Lipschitz* if $\text{Lip}(f) \leq k$, and that it is *Lipschitz* if it is k-Lipschitz for some $k \geq 0$. We denote the set of all k-Lipschitz functions on X with respect to d by $\text{Lip}_k(X, d)$, and that of all Lipschitz functions on X with respect to d by $\text{Lip}(X, d)$. It is plain that $\text{Lip}_k(X, d) \subseteq \text{Lip}(X, d) \subseteq \mathbf{C}(X)$ for any $k \geq 0$.

We will be particularly interested in 1-Lipschitz maps here, and always view $\text{Lip}_1(X, d)$ as a (metric) subspace of $\mathbf{C}(X)$. As such, it is easy to see that $\text{Lip}_1(X, d)$ is closed in $\mathbf{C}(X)$. (Indeed, if (f_m) is a sequence in $\mathcal{F}$ such that $f_m \to f$ for some $f \in \mathbf{C}(X)$, then, for any x and y in X, and any positive integer m,

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_m(x)| + |f_m(x) - f_m(y)| + |f_m(y) - f(y)| \\ &\leq |f(x) - f_m(x)| + d(x, y) + |f_m(y) - f(y)|, \end{aligned}$$

so letting $m \to \infty$, we find $|f(x) - f(y)| \leq d(x,y)$, that is, $f \in \mathrm{Lip}_1(X,d)$.) It is also readily verified that $\mathrm{Lip}_1(X,d)$ is absolutely convex in $\mathbf{C}(X)$.[8] Finally, $\mathrm{Lip}_1(X,d)$ is a sublattice of $\mathbf{C}(X)$. Indeed, for any real numbers α, β, γ and δ, we have

$$|\max\{\alpha,\beta\} - \max\{\gamma,\delta\}| \leq \max\{|\alpha-\gamma|, |\beta-\delta|\},$$

and hence

$$|\max\{f(x),g(x)\} - \max\{f(y),g(y)\}| \leq \max\{|f(x)-f(y)|, |g(x)-g(y)|\} \leq d(x,y)$$

for every $f, g \in \mathrm{Lip}_1(X,d)$ and $x, y \in X$. Thus, $\max\{f,g\} \in \mathrm{Lip}_1(X,d)$ for any $f, g \in \mathrm{Lip}_1(X,d)$. As $|\min\{\alpha,\beta\} - \min\{\gamma,\delta\}|$ is also bounded above by $\max\{|\alpha-\gamma|, |\beta-\delta|\}$ for any real numbers α, β, γ and δ, this reasoning also shows that $\min\{f,g\} \in \mathrm{Lip}_1(X,d)$ for any $f, g \in \mathrm{Lip}_1(X,d)$. In sum, we have the following well-known fact which we will utilize in what follows.

Fact. $\mathrm{Lip}_1(X,d)$ is a closed and absolutely convex sublattice of $\mathbf{C}(X)$ for any metric space (X,d).

Each of the following lemmata will be used in the proof of Theorem 2a.

Lemma 1. Let (X,d) be a metric space.

(a) For every (pairwise) distinct $x_1, x_2, x_3 \in X$, there is a map $u \in \mathrm{Lip}_1(X,d)$ such that $u(x_1) > u(x_2) > u(x_3)$.

(b) For every (pairwise) distinct $x_1, x_2, x_3, x_4 \in X$, there is a map $u \in \mathrm{Lip}_1(X,d)$ such that $u(x_1) > u(x_2)$ and $u(x_3) > u(x_4)$.

Proof. (a) Define the real map u on X as

$$u := \tfrac{1}{2}\min\{d(\cdot,x_3), d(\cdot,x_2)\} - \tfrac{1}{2}\min\{d(\cdot,x_1), d(\cdot,x_2)\}.$$

As is well-known, and is easily deduced from the triangle inequality for d, the map $d(\cdot,y)$ is 1-Lipschitz on (X,d) for any $y \in X$. By the Fact noted above, therefore, $u \in \mathrm{Lip}_1(X,d)$. Moreover,

$$u(x_1) = \tfrac{1}{2}\min\{d(x_1,x_3), d(x_1,x_2)\} > 0 = u(x_2) = 0 > -\tfrac{1}{2}\min\{d(x_3,x_1), d(x_3,x_2)\} = u(x_3).$$

(b) Define the real map u on X as $u := \min\{d(\cdot,x_2), d(\cdot,x_4)\}$. Once again, $u \in \mathrm{Lip}_1(X,d)$ by the Fact noted above, while $u(x_1) > 0 = u(x_2)$ and $u(x_3) > 0 = u(x_4)$. ■

Lemma 2. Let X be a compact metrizable space and d any distance function on X that metrizes X. Then,

$$\{f \in \mathrm{Lip}_1(X,d) : f(x^*) = 0\} \tag{13}$$

is a compact subset of $\mathbf{C}(X)$ for any x^* in X.

Proof. Fix an arbitrary x^* in X, and denote the set in (13) by $\mathcal{F}$. It is plain that $\{f \in \mathbf{C}(X) : f(x^*) = 0\}$ is closed in $\mathbf{C}(X)$, while we have already verified above that $\mathrm{Lip}_1(X,d)$ is a closed subset of $\mathbf{C}(X)$. Being the intersection of these two sets, therefore, $\mathcal{F}$ is closed in $\mathbf{C}(X)$. Furthermore,

$$\sup_{f\in\mathcal{F}} |f(x)| = \sup_{f\in\mathcal{F}} |f(x) - f(x^*)| \leq d(x,x^*),$$

[8] A subset S of a (real) linear space X is said to be *balanced* if $\lambda S \subseteq S$ for any real number λ with $|\lambda| \leq 1$. In turn, S is said to be *absolutely convex* if it is balanced and convex.

so $\{f(x) : f \in \mathcal{F}\}$ is bounded in $\mathbb{R}$ for each $x \in X$. Conclusion: $\mathcal{F}$ is pointwise bounded. On the other hand, it is easy to check that any family of Lipschitz maps whose Lipschitz constants is bounded from above is equicontinuous. In particular, $\mathrm{Lip}_1(X, d)$, and hence $\mathcal{F}$, is equicontinuous. We may then apply the Arzelá-Ascoli Theorem to conclude that $\mathcal{F}$ is compact in $\mathbf{C}(X)$. ■

In what follows, for any metric space (X, d), we denote the *closed ε-ball* around a point x in X by $B(x, \varepsilon)$. (The dependence of this set on the metric of X is not made explicit in our notation, but this will not cause any confusion.) Put precisely, for any $\varepsilon > 0$ and $x \in X$, we have $B(x, \varepsilon) := \{\omega \in X : d(\omega, x) \leq \varepsilon\}$.

Lemma 3. Let X be a metrizable space and d any distance function on X that metrizes X. Take any $f \in \mathrm{Lip}_1(X, d)$ and $x^* \in X$ with $f(x^*) = 0$. Then, for any $\varepsilon > 0$, there is a $g \in \mathrm{Lip}_1(X, d)$ such that

$$d_\infty(f, g) \leq \varepsilon, \qquad g|_{B(x^*,\varepsilon)} = 0 \qquad \text{and} \qquad \mathrm{sgn} f|_{X\backslash B(x^*,\varepsilon)} = \mathrm{sgn}\, g|_{X\backslash B(x^*,\varepsilon)}.$$

Proof. Fix any $\varepsilon \in (0, 1)$, and define

$$S := \{\omega \in X\backslash B(x^*, \varepsilon) : |f(\omega)| \leq d(\omega, x^*) - \varepsilon\}$$

and

$$T := \{\omega \in X\backslash B(x^*, \varepsilon) : |f(\omega)| > d(\omega, x^*) - \varepsilon\}.$$

Obviously, $\{B(x^*, \varepsilon), S, T\}$ is a partition of X. We define $g : X \to \mathbb{R}$ by

$$g := f\mathbf{1}_S + ((d(\cdot, x^*) - \varepsilon)\mathrm{sgn} f)\, \mathbf{1}_T.$$

(Here, for any subset A of X, by $\mathbf{1}_A$ we mean the indicator function of A on X.) Obviously, $g|_{B(x^*,\varepsilon)} = 0$ and $\mathrm{sgn} f|_{X\backslash B(x^*,\varepsilon)} = \mathrm{sgn}\, g|_{X\backslash B(x^*,\varepsilon)}$. On the other hand, $|f(\omega)| = |f(\omega) - f(x^*)|$, and hence, as f is 1-Lipschitz, we have

$$|f(\omega)| \leq d(\omega, x^*) \qquad \text{for every } \omega \in X.$$

It follows that $|f(\omega)| \leq \varepsilon$ for any $\omega \in B(x^*, \varepsilon)$, while, again by this inequality,

$$\begin{aligned} |f(\omega) - g(\omega)| &= ||f(\omega)|\,\mathrm{sgn} f(\omega) - (d(\omega, x^*) - \varepsilon)\mathrm{sgn} f(\omega)| \\ &= |f(\omega)| - (d(\omega, x^*) - \varepsilon) \\ &\leq \varepsilon \end{aligned}$$

for any $\omega \in T$. Thus: $d_\infty(f, g) \leq \varepsilon$. It remains to prove that $g \in \mathrm{Lip}_1(X, d)$.

Take any y and z in X. We wish to prove that

$$|g(y) - g(z)| \leq d(y, z). \tag{14}$$

This inequality obviously holds when both y and z belong to $B(x^*, \varepsilon)$, while the fact that f is 1-Lipschitz implies that it also holds when both y and z belong to S. Suppose, then, both y and z are in T. In this case, if $\mathrm{sgn} f(y) = \mathrm{sgn} f(z)$, we have

$$|g(y) - g(z)| = |d(y, x^*) - d(z, x^*)| \leq d(y, z).$$

Otherwise, say, if $\mathrm{sgn} f(y) > \mathrm{sgn} f(z)$, then $f(y) > 0$ and $f(z) < 0$ – we cannot have $f(z) = 0$ as $z \in T$ – so $f(y) > d(y, x^*) - \varepsilon$ and $-f(z) > d(z, x^*) - \varepsilon$, whence

$$\begin{aligned} |g(y) - g(z)| &= |d(y, x^*) - \varepsilon + d(z, x^*) - \varepsilon| \\ &= (d(y, x^*) - \varepsilon) + (d(z, x^*) - \varepsilon) \\ &< f(y) - f(z) \\ &\leq d(y, z) \end{aligned}$$

because f is 1-Lipschitz. We have thus proved that (14) is valid so long as y and z belong to the same cell of the partition $\{B(x^*, \varepsilon), S, T\}$.

Let us now consider the case where y and z belong to different cells of $\{B(x^*, \varepsilon), S, T\}$. First, suppose one of them, say, y, belongs to $B(x^*, \varepsilon)$ while $z \in S \cup T$. If $z \in S$, then,

$$|g(y) - g(z)| = |g(z)| = |f(z)| \leq d(z, x^*) - \varepsilon,$$

and if $z \in T$, then

$$|g(y) - g(z)| = |g(z)| = |(d(z, x^*) - \varepsilon)\mathrm{sgn} f(z)| = d(z, x^*) - \varepsilon.$$

But, as $y \in B(x^*, \varepsilon)$,

$$d(z, x^*) - \varepsilon \leq d(y, z) + (d(x^*, y) - \varepsilon) \leq d(y, z),$$

and combining these findings show that (14) is valid so long as $y \in B(x^*, \varepsilon)$ and $z \in S \cup T$. Finally, we consider the case where an element of $\{y, z\}$ belongs to S and the other to T. Suppose, then, $(y, z) \in S \times T$. In this case,

$$|g(y) - g(z)| = \begin{cases} |f(y) - (d(z, x^*) - \varepsilon)|, & \text{if } f(z) \geq 0 \\ |f(y) + (d(z, x^*) - \varepsilon)|, & \text{otherwise.} \end{cases}$$

But, as $y \in S$,

$$f(y) - (d(z, x^*) - \varepsilon) \leq d(y, x^*) - \varepsilon - (d(z, x^*) - \varepsilon) \leq d(y, z)$$

while, when $f(z) \geq 0$, the facts that z is in T and f is 1-Lipschitz imply

$$(d(z, x^*) - \varepsilon) - f(y) \leq f(z) - f(y) \leq d(y, z).$$

Similarly, as $y \in S$,

$$-f(y) - (d(z, x^*) - \varepsilon) \leq d(y, x^*) - \varepsilon - (d(z, x^*) - \varepsilon) \leq d(y, z)$$

while, when $f(z) < 0$, the facts that z is in T and f is 1-Lipschitz imply

$$f(y) + (d(z, x^*) - \varepsilon) \leq f(y) - f(z) \leq d(y, z).$$

We thus conclude that (14) is valid whenever $(y, z) \in S \times T$ as well. This concludes our proof. ■

Proof of Theorem 2a

Given any Hausdorff topological space X, and any x and y in X, we define the binary relation $\succsim_{xy}$ on X as

$$\succsim_{xy} := \{(x, y)\} \cup \triangle_X.$$

It is plain that any such binary relation is a continuous preorder on X. Moreover, when X is a metric space, such a preorder is sure to possess a multi-utility representation by means of 1-Lipschitz functions, as we show next.

Lemma 4. Let (X, d) be a metric space, and $x, y \in X$. Then, there is an $\mathcal{U} \subseteq \mathrm{Lip}_1(X, d)$ such that

$$a \succsim_{xy} b \quad \text{iff} \quad \mathcal{U}(a) \geq \mathcal{U}(b) \tag{15}$$

for every a and b in X.

Proof. If $x = y$, then $\succsim_{xy} = \triangle_X$, and setting $\mathcal{U} := \{d(\cdot, z) : z \in X\}$ completes the proof. We assume, then, $x \neq y$, and put

$$\mathcal{U} := \{u \in \mathrm{Lip}_1(X, d) : u(x) > u(y)\}.$$

Take any a and b in X and note that the "only if" part of (15) is an immediate consequence of the definitions of $\succsim_{xy}$ and $\mathcal{U}$. Conversely, assume that $a \succsim_{xy} b$ does not hold. Then, clearly, $a \neq b$ and $(a,b) \neq (x,y)$. There are three cases to consider. First, suppose $|\{x,y,a,b\}| = 2$ so that $a = y$ and $b = x$. But then, where $u := d(\cdot,y) \in \mathrm{Lip}_1(X,d)$, we have $u(b) = u(x) > u(y) = u(a)$, so $\mathcal{U}(a) \geq \mathcal{U}(b)$ does not hold. Second, suppose $|\{x,y,a,b\}| = 3$. In this case, either $x \in \{a,b\}$ or (exclusive) $y \in \{a,b\}$. If $x = a$, we use Lemma 3.(a) to find a $u \in \mathrm{Lip}_1(X,d)$ with $u(b) > u(x) > u(y)$, while if $x = b$, we use Lemma 3.(a) to find a $u \in \mathrm{Lip}_1(X,d)$ with $u(b) = u(x) > u(y) > u(a)$. Thus, $\mathcal{U}(a) \geq \mathcal{U}(b)$ does not hold when $x \in \{a,b\}$. That this is also the case when $y \in \{a,b\}$ is similarly established by using Lemma 3.(a). Third, suppose $|\{x,y,a,b\}| = 4$. In this case, we use Lemma 3.(b) to find a $u \in \mathcal{U}$ with $u(b) > u(a)$, so, again, $\mathcal{U}(a) \geq \mathcal{U}(b)$ does not hold, and our proof is complete. ■

In what follows, for any topological space Z, by $\mathbf{k}(Z)$ we mean the set of all nonempty compact subsets of Z. When Z is a metric space, we understand that $\mathbf{k}(Z)$ is metrized by means of the Hausdorff metric. We recall that if Z is compact, then $\mathbf{k}(Z)$ too is compact. Similarly, it is well-known that separability of Z implies that of $\mathbf{k}(Z)$.

Proof of Theorem 2a. Suppose there is a $\mathbb{U} \in \mathbf{k}(\mathbf{k}(\mathbf{C}(X)))$ such that (6) holds for each $x, y \in X$. Then, obviously, $\mathbf{R}$ is reflexive. To show that it is also continuous, take any sequence (x_m, y_m) in $\mathbf{R}$ that converges to some $(x,y) \in X \times X$. Then, for each positive integer m, there is an $\mathcal{U}_m \in \mathbb{U}$ such that $u(x_m) \geq u(y_m)$ for each $u \in \mathcal{U}_m$. On the other hand, as $\mathbb{U}$ is compact, there is a strictly increasing sequence (m_k) of positive integers such that $\mathcal{U}_{m_k} \to^{\mathrm{H}} \mathcal{U}$ for some $\mathcal{U} \in \mathbb{U}$. Now take an arbitrary u in $\mathcal{U}$ and notice that there must be a sequence (u_{m_k}) in $\mathcal{U}_{m_1} \times \mathcal{U}_{m_2} \times \cdots$ such that $\|u_{m_k} - u\|_\infty \to 0$. Then, as

$$\begin{aligned} |u_{m_k}(x_{m_k}) - u(x)| &\leq |u_{m_k}(x_{m_k}) - u(x_{m_k})| + |u(x_{m_k}) - u(x)| \\ &\leq \|u_{m_k} - u\|_\infty + |u(x_{m_k}) - u(x)|, \end{aligned}$$

for each k, we have $u_{m_k}(x_{m_k}) \to u(x)$ and similarly, $u_{m_k}(y_{m_k}) \to u(y)$. Consequently, since $u_{m_k}(x_{m_k}) \geq u_{m_k}(y_{m_k})$ for each k, we find $u(x) \geq u(y)$. In view of the arbitrary choice of u, therefore, we conclude that $\min_{u \in \mathcal{U}}(u(x) - u(y)) \geq 0$, which means (x,y) belongs to $\mathbf{R}$, as we sought. Thus: $\mathbf{R}$ is a continuous and reflexive binary relation on X.

Conversely, let $\mathbf{R}$ be a continuous and reflexive binary relation on X, and denote the metric of X by d. As in the proof of Theorem 1, fix an arbitrary (x,y) in $X \times X$, and put $\succsim_{xy} := \{(x,y)\} \cup \triangle_X$. By Lemma 4, there is an $\mathcal{V}_{xy} \subseteq \mathrm{Lip}_1(X,d)$ such that

$$a \succsim_{xy} b \quad \text{iff} \quad \mathcal{V}_{xy}(a) \geq \mathcal{V}_{xy}(b)$$

for each $a, b \in X$. Put

$$\mathcal{U}_{xy} := \{u \in \mathrm{Lip}_1(X,d) : u(x) \geq u(y) = 0\}.$$

It is obvious that $\mathcal{U}_{xy}$ is a closed subset of $\{u \in \mathrm{Lip}_1(X,d) : u(y) = 0\}$, so by Lemma 2, $\mathcal{U}_{xy} \in \mathbf{k}(\mathbf{C}(X))$. Moreover, for any a and b in X, it follows readily from the definition of $\succsim_{xy}$ that $a \succsim_{xy} b$ implies $\mathcal{U}_{xy}(a) \geq \mathcal{U}_{xy}(b)$, while, conversely, $\mathcal{U}_{xy}(a) \geq \mathcal{U}_{xy}(b)$ implies $\mathcal{V}_{xy}(a) \geq \mathcal{V}_{xy}(b)$ (because $\{v - v(y) : v \in \mathcal{V}_{xy}\} \subseteq \mathcal{U}_{xy}$), and hence, $a \succsim_{xy} b$. Thus:

$$a \succsim_{xy} b \quad \text{iff} \quad \inf_{u \in \mathcal{U}_{xy}} (u(a) - u(b)) \geq 0$$

for each $a, b \in X$. But then, since $\mathbf{R} = \bigcup\{\succsim_{xy} : (x,y) \in \mathbf{R}\}$, setting $\mathbb{U} := \{\mathcal{U}_{xy} : (x,y) \in \mathbf{R}\}$ yields

$$a\ \mathbf{R}\ b \quad \text{iff} \quad \max_{\mathcal{U} \in \mathbb{U}} \inf_{u \in \mathcal{U}} (u(a) - u(b)) \geq 0$$

for each $a, b \in X$.[9] But as each $\mathcal{U} \in \mathbb{U}$ is compact, $\{u(\omega) : u \in \mathcal{U}\}$ is a compact set of real numbers for any $\omega \in X$ and $\mathcal{U} \in \mathbb{U}$. It follows that we can in fact write

$$a \mathbf{R} b \quad \text{iff} \quad \max_{\mathcal{U} \in \mathbb{U}} \min_{u \in \mathcal{U}} (u(a) - u(b)) \geq 0.$$

Consequently, our proof will be complete if we can show that $\mathbb{U}$ is a compact subset of $\mathbf{k}(\mathbf{C}(X))$.

Let (x_m, y_m) be any sequence in $\mathbf{R}$. But $\mathbf{R}$ is a compact subset of $X \times X$, because X is compact, and $\mathbf{R}$ is closed in $X \times X$. Consequently, there is a subsequence of (x_m, y_m) that converges in $\mathbf{R}$. We denote this subsequence also by (x_m, y_m) to simplify the notation, and note that $x := \lim x_m \mathbf{R} \lim y_m =: y$. The task at hand is to show that

$$\mathcal{U}_{x_m y_m} \to^{\mathrm{H}} \mathcal{U}_{xy}.$$

To this end, fix an arbitrary $\varepsilon > 0$, and put

$$\delta := \begin{cases} \min\{\varepsilon, d(x, y)\}, & \text{if } x \neq y \\ \varepsilon, & \text{otherwise.} \end{cases}$$

As $(x_m, y_m) \to (x, y)$, we can choose a positive integer M such that

$$\max\{d(x_m, x), d(y_m, y)\} < \tfrac{\delta}{6} \quad \text{for each } m \geq M.$$

Now take any $u \in \mathcal{U}_{xy}$, and apply Lemma 3 to u to find a map $u_1 \in \mathrm{Lip}_1(X, d)$ such that

$$\|u - u_1\|_\infty \leq \tfrac{\delta}{3}, \quad u_1|_{B(y,\delta/3)} = 0 \quad \text{and} \quad \mathrm{sgn} u|_{X \setminus B(y,\delta/3)} = \mathrm{sgn} u_1|_{X \setminus B(y,\delta/3)}. \tag{16}$$

One useful implication of the latter two of these properties is that

$$u_1(x) \geq 0. \tag{17}$$

(Indeed, if $x \in B(y, \delta/3)$, we have $u_1(x) = 0$, and otherwise, $\mathrm{sgn} u_1(x) = \mathrm{sgn} u(x)$, while, as $u \in \mathcal{U}_{xy}$, we have $u(x) \geq u(y) = 0$.) Finally, we apply Lemma 3 to $u_1 - u_1(x)$ to find a map $u_2 \in \mathrm{Lip}_1(X, d)$ such that

$$\|(u_1 - u_1(x)) - u_2\|_\infty \leq \tfrac{\delta}{3}, \quad u_2|_{B(x,\delta/3)} = 0 \tag{18}$$

and

$$\mathrm{sgn}(u_1 - u_1(x))|_{X \setminus B(x,\delta/3)} = \mathrm{sgn} u_2|_{X \setminus B(x,\delta/3)}. \tag{19}$$

Let us fix an arbitrary integer $m \geq M$. As $x_m \in B(x, \delta/3)$, we have $u_2(x_m) = 0$ by (18). Similarly, $u_1(y_m) = 0$. Now, if $y_m \in B(x, \delta/3)$, we have $u_2(y_m) = 0$ by (18). On the other hand, if $y_m \in X \setminus B(x, \delta/3)$, we have $\mathrm{sgn} u_2(y_m) = \mathrm{sgn}(u_1(y_m) - u_1(x)) = \mathrm{sgn}(-u_1(x)) \leq 0$ by (17). We thus find here that

$$u_2(x_m) = 0 \geq u_2(y_m).$$

Let us now set $u_{3,m} := u_2 - u_2(y_m)$, and note that $u_{3,m}(x_m) \geq 0 = u_{3,m}(y_m)$, which means $u_{3,m} \in \mathcal{U}_{x_m y_m}$. Moreover, as $u_1(y_m) = 0$, we have

$$\begin{aligned} \|u_{3,m} - u_1\|_\infty &= \|u_{3,m} + u_2(y_m) - (u_1 - u_1(x)) - u_2(y_m) - u_1(x)\|_\infty \\ &\leq \|u_2 - (u_1 - u_1(x))\|_\infty + |u_2(y_m) - (u_1(y_m) - u_1(x))| \\ &\leq \tfrac{\delta}{3} + \|u_2 - (u_1 - u_1(x))\|_\infty \\ &\leq \tfrac{2\delta}{3} \end{aligned}$$

[9]Given the structure of $\mathbb{U}$, we can show that $a \mathbf{R} b$ implies $\inf_{u \in \mathcal{U}}(u(a) - u(b)) \geq 0 \geq \inf_{u \in \mathcal{U}'}(u(a) - u(b))$ for some $\mathcal{U} \in \mathbb{U}$ and for all $\mathcal{U}' \in \mathbb{U}$, whereas not $a \mathbf{R} b$ implies $\inf_{u \in \mathcal{U}}(u(a) - u(b)) = -d(a, b)$ for any $\mathcal{U} \in \mathbb{U}$. So, the value $\max_{\mathcal{U} \in \mathbb{U}} \inf_{u \in \mathcal{U}}(u(a) - u(b))$ is well-defined without invoking compactness of $\mathbb{U}$.

by (18). Consequently, by (16),

$$d_\infty(u,\mathcal{U}_{x_m y_m}) \le \|u - u_{3,m}\|_\infty \le \|u - u_1\|_\infty + \|u_1 - u_{3,m}\|_\infty \le \delta.$$

As $\delta \le \varepsilon$, therefore, we proved that $d_\infty(u,\mathcal{U}_{x_m y_m}) \le \varepsilon$ for each $m \ge M$ and $u \in \mathcal{U}_{xy}$. In other words,

$$\sup_{u\in\mathcal{U}_{xy}} d_\infty(u,\mathcal{U}_{x_m y_m}) \le \varepsilon \quad \text{for each } m \ge M.$$

Besides, by exchanging the roles of $\mathcal{U}_{xy}$ and $\mathcal{U}_{x_m y_m}$ in the above argument, we can similarly prove that

$$\sup_{u\in\mathcal{U}_{x_m y_m}} d_\infty(u,\mathcal{U}_{xy}) \le \varepsilon \quad \text{for each } m \ge M.$$

It follows that $d^H(\mathcal{U}_{xy},\mathcal{U}_{x_m y_m}) \le \varepsilon$ for each $m \ge M$. As $\varepsilon > 0$ is arbitrary here, we conclude that $\mathcal{U}_{x_m y_m} \to^{\mathrm{H}} \mathcal{U}_{xy}$, and hence that $\mathbb{U}$ is compact in $\mathbf{k}(\mathbf{C}(X))$. This completes the proof of Theorem 2a. ■

Proof of Theorem 2b

We will use the following technical lemma in the substance of the proof.

Lemma 5. Let X be a compact metric space, and define the function $\Phi : \mathbf{k}(\mathbf{C}(X)) \times X \times X \to \mathbb{R}$ by

$$\Phi(\mathcal{U}, x, y) := \sup_{u\in\mathcal{U}}(u(x) - u(y)).$$

Then, Φ is continuous and

$$\Phi(\mathcal{U}, x, y) = \max_{u\in\mathcal{U}}(u(x) - u(y)) \tag{20}$$

for every $\mathcal{U} \in \mathbf{k}(\mathbf{C}(X))$ and $x, y \in X$.

Proof. As uniform convergence implies pointwise convergence, for any $x, y \in X$, the map $u \mapsto u(x) - u(y)$ is continuous on any given $\mathcal{U} \in \mathbf{k}(\mathbf{C}(X))$. Thus, the fact that (20) holds follows from a straightforward application of the Weierstrass Theorem. To prove the continuity of Φ, put $\Theta := \mathbf{k}(\mathbf{C}(X)) \times X \times X$, and define the correspondence $\Gamma : \Theta \rightrightarrows \mathbf{C}(X)$ by $\Gamma(\mathcal{U}, x, y) := \mathcal{U}$. It is plain that Γ is a compact-valued and continuous correspondence. Next, consider the real map φ defined on $\mathbf{C}(X) \times \Theta \to \mathbb{R}$ by

$$\varphi(u, (\mathcal{U}, x, y)) := u(x) - u(y).$$

To see that this map is continuous, let $(u_m, (\mathcal{U}_m, x_m, y_m))$ be a convergent sequence in $\mathbf{C}(X) \times \Theta$, and denote the limit of this sequence by $(u, (\mathcal{U}, x, y))$. We wish to show that $u_m(x_m) - u_m(y_m) \to u(x) - u(y)$. By symmetry, it is enough to show that $u_m(x_m) \to u(x)$ here. To this end, fix an arbitrary $\varepsilon > 0$, and note that, as $\{u, u_1, u_2, ...\}$ is compact, and hence equicontinuous (at x) in $\mathbf{C}(X)$, we can find a $\delta > 0$ such that $\sup_{i\in\mathbb{N}} |u_i(\omega) - u_i(x)| \le \varepsilon/2$ for any $\omega \in B(x, \delta)$. But then, choosing M to be a positive integer large enough so that $d(x_m, x) < \delta$ and $\|u_m - u\| \le \varepsilon/2$ whenever $m \ge M$, we find

$$|u_m(x_m) - u(x)| \le |u_m(x_m) - u_m(x)| + |u_m(x) - u(x)| \le \varepsilon$$

for each $m \ge M$. Given the arbitrariness of $\varepsilon > 0$, then, we conclude that φ is continuous. Given these observations on Γ and φ, we may apply Berge's Maximum Theorem to conclude that the map $(\mathcal{U}, x, y) \mapsto \max_{u\in\Gamma(\mathcal{U},x,y)} \varphi(u, (\mathcal{U}, x, y))$ is continuous on Θ. As this map is none other than Φ, we are done. ■

Proof of Theorem 2b. To see the "if" part of the assertion, suppose there is a nonempty $\mathbb{V} \in 2^{\mathbf{k}(\mathbf{C}(X))}$ as in the statement of the theorem. We need to show that $\mathbf{R}$ is continuous.

To this end, take any sequence (x_m, y_m) in $\mathbf{R}$ that converges to some $(x, y) \in X \times X$. By the choice of $\mathbb{V}$, for every $m \in \mathbb{N}$ and $\mathcal{V} \in \mathbb{V}$ we have $\max_{v\in\mathcal{V}}(v(x_m) - v(y_m)) \geq 0$. But then, by Lemma 5,

$$\inf_{\mathcal{V}\in\mathbb{V}} \max_{v\in\mathcal{V}}(v(x) - v(y)) = \inf_{\mathcal{V}\in\mathbb{V}} \lim_{m\to\infty} \max_{v\in\mathcal{V}}(v(x_m) - v(y_m)) \geq 0,$$

which means that $x \mathbf{R} y$, as we sought.

Conversely, let $\mathbf{R}$ be a continuous and reflexive binary relation on X, and use Theorem 2a to find a $\mathbb{U} \in \mathbf{k}(\mathbf{k}(\mathbf{C}(X)))$ such that

$$x \mathbf{R} y \quad \text{iff} \quad [\text{there is a } \mathcal{U} \in \mathbb{U} \text{ such that } u(x) \geq u(y) \text{ for each } u \in \mathcal{U}]$$

for every x and y in X. We proceed in the way we have obtained Theorem 1b from Theorem 1a. Fix any $(x, y) \in X^2 \backslash \mathbf{R}$, define $\xi_{x,y} : \mathbf{k}(\mathbf{C}(X)) \to \mathbb{R}$ by

$$\xi_{x,y}(\mathcal{U}) := \max_{u\in\mathcal{U}}(u(y) - u(x)),$$

and note that $\xi_{x,y}$ is a (well-defined) continuous map (Lemma 5). This map is $\mathbb{R}_{++}$-valued on $\mathbb{U}$. Indeed, since $x \mathbf{R} y$ is false, for any $\mathcal{U} \in \mathbb{U}$ and there is some $u \in \mathcal{U}$ with $u(y) > u(x)$, whence $\xi_{x,y}(\mathcal{U}) \geq u(y) - u(x) > 0$. Thus: $\xi_{x,y}(\mathcal{U}) > 0$ for each $\mathcal{U} \in \mathbb{U}$. On the other hand, as $\mathbb{U}$ is compact and $\xi_{x,y}$ is continuous, $\xi_{x,y}$ attains its minimum on $\mathbb{U}$. As this map $\mathbb{R}_{++}$-valued on $\mathbb{U}$, therefore,

$$\xi_{x,y} := \min_{\mathcal{U}\in\mathbb{U}} \xi_{x,y}(\mathcal{U}) > 0.$$

We now define

$$\mathcal{V}_{x,y} := \{v \in \bigcup \mathbb{U} : v(y) - v(x) \geq \xi_{x,y}\}.$$

Several observations about this collection are in order. First, $\mathcal{V}_{x,y}$ is compact. Indeed, as the compact union of compact sets is compact – see Theorem 2.5 of Michael (1951) – $\bigcup \mathbb{U}$ is a compact subset of $\mathbf{C}(X)$. As it is easily verified that the set of all $v \in \mathbf{C}(X)$ with $v(y) - v(x) \geq \xi_{x,y}$ is closed in $\mathbf{C}(X)$, therefore, $\mathcal{V}_{x,y}$ is a compact subset of $\mathbf{C}(X)$. Second,

$$v(y) > v(x) \quad \text{for each } v \in \mathcal{V}_{x,y},$$

because $\xi_{x,y} > 0$. Third,

$$\mathcal{U} \cap \mathcal{V}_{x,y} \neq \emptyset \quad \text{for each } \mathcal{U} \in \mathbb{U}.$$

Indeed, if $\mathcal{U} \in \mathbb{U}$, we have $\max_{u\in\mathcal{U}}(u(y) - u(x)) = \xi_{x,y}(\mathcal{U}) \geq \xi_{x,y}$, which means that there is some $u \in \mathcal{U}$ with $u(y) - u(x) \geq \xi_{x,y}$.

To complete our proof, we define

$$\mathbb{W} := \{\mathcal{V}_{x,y} : (x, y) \in X^2 \backslash \mathbf{R}\}.$$

In view of what we have found in the previous paragraph, it is plain that $\mathbb{W}$ is a collection of compact subsets of $\mathbf{C}(X)$ such that (i) for any $(x, y) \in X^2 \backslash \mathbf{R}$, there is a $\mathcal{V} \in \mathbb{W}$ such that $v(y) > v(x)$ for each $v \in \mathcal{V}$, and (ii) $\mathcal{U} \cap \mathcal{V} \neq \emptyset$ for each $(\mathcal{U}, \mathcal{V}) \in \mathbb{U} \times \mathbb{W}$. Now take any x and y in X. We claim:

$$x \mathbf{R} y \quad \text{iff} \quad \max_{v\in\mathcal{V}}(v(x) - v(y)) \geq 0 \text{ for each } \mathcal{V} \in \mathbb{W}. \tag{21}$$

To see this, suppose first that $x \mathbf{R} y$ is false. Then, by (i),

$$0 > \max_{v\in\mathcal{V}_{x,y}}(v(x) - v(y)) \geq \inf_{\mathcal{V}\in\mathbb{W}} \max_{v\in\mathcal{V}}(v(x) - v(y)).$$

Conversely, suppose now that $x \mathbf{R} y$. Then, by the choice of $\mathbb{U}$, there is a $\mathcal{U} \in \mathbb{U}$ such that $u(x) \geq u(y)$ for each $u \in \mathcal{U}$. But then, by (ii), for every $\mathcal{V}$ in $\mathbb{W}$ we have $v(x) \geq v(y)$ for some $v \in \mathcal{V}$, whence $\max_{v\in\mathcal{V}}(v(x) - v(y)) \geq 0$.

Now, it is without loss of generality to assume that $\mathbb{W}$ does not contain the singleton $\{\mathbf{0}\}$, where $\mathbf{0}$ stands for the zero function $\mathbf{0}$ on X.[10] Moreover, as X is compact, $\mathbf{C}(X)$ is separable. It follows that $\mathbf{k}(\mathbf{C}(X))$ is a separable metric space, and hence, so is $\mathbb{W}$. Pick any countable dense subset of $\mathbb{W}$, and enumerate this set as $\{\mathcal{V}_1, \mathcal{V}_2, ...\}$. As $\mathcal{V} \mapsto \max_{v \in \mathcal{V}}(v(x) - v(y))$ is a continuous map on $\mathbf{k}(\mathbf{C}(X))$ (Lemma 5), it follows from (21) that

$$x \mathbf{R}\, y \quad \text{iff} \quad \max_{v \in \mathcal{V}_i}(v(x) - v(y)) \geq 0 \text{ for each } i = 1, 2, ...$$

for every $x, y \in X$. On the other hand, for each positive integer i, $\mathcal{V}_i$ is a compact, and hence bounded, subset of $\mathbf{C}(X)$, and hence $K_i := \sup\{\|v\|_\infty : v \in \mathcal{V}_i\}$ is a real number. As none of the $\mathcal{V}_i$s equals $\{\mathbf{0}\}$, we have $K_i > 0$ for each i. Furthermore, obviously,

$$x \mathbf{R}\, y \quad \text{iff} \quad \max_{v \in \mathcal{V}_i}\left(\frac{v(x)}{iK_i} - \frac{v(y)}{iK_i}\right) \geq 0 \text{ for each } i = 1, 2, ...$$

for every $x, y \in X$. Now define $\mathbb{V} := \{\{\mathbf{0}\}, \frac{1}{K_1}\mathcal{V}_1, \frac{1}{2K_2}\mathcal{V}_2, \frac{1}{3K_3}\mathcal{V}_3, ...\}$, and note that

$$x \mathbf{R}\, y \quad \text{iff} \quad \inf_{\mathcal{V} \in \mathbb{V}} \max_{v \in \mathcal{V}} (v(x) - v(y)) \geq 0.$$

But it is readily checked that $\frac{1}{mK_m}\mathcal{V}_m \to \{\mathbf{0}\}$ relative to the Hausdorff metric, and hence $\mathbb{V}$ is compact in $\mathbf{k}(\mathbf{C}(X))$. In view of the continuity of the map $\mathcal{V} \mapsto \max_{v \in \mathcal{V}}(v(x) - v(y))$, therefore, we arrive at (7), completing our proof. ■

Proof of Theorem 3

The "if" part of this theorem follows readily from that of Theorem 2a. Its "only if" part, on the other hand, is an immediate consequence of the following result.

Lemma 6. Let X be a compact metrizable space and $\succsim$ a continuous preorder on X. Then, there is a distance function d on X that metrizes X, and a compact subset $\mathcal{U}$ of $\mathbf{C}(X)$ such that $\mathcal{U} \subseteq \mathrm{Lip}_1(X, d)$ and

$$x \succsim y \quad \text{iff} \quad \mathcal{U}(x) \geq \mathcal{U}(y)$$

for every x and y in X.

Proof. By Corollary 2 of Evren and Ok (2011), there is an $\mathcal{U}^0 \subseteq \mathbf{C}(X)$ such that

$$x \succsim y \quad \text{iff} \quad \mathcal{U}^0(x) \geq \mathcal{U}^0(y)$$

for every x and y in X. (Without loss of generality, we assume that the zero function on X does not belong to $\mathcal{U}^0$.) As X is compact, $\mathbf{C}(X)$, and hence $\mathcal{U}^0$, is separable, so there is a countable dense subset $\mathcal{U}^1$ of $\mathcal{U}^0$. It is easily verified that we also have

$$x \succsim y \quad \text{iff} \quad \mathcal{U}^1(x) \geq \mathcal{U}^1(y)$$

for every x and y in X. Given the countability of $\mathcal{U}^1$, we may invoke Theorem 3, and then Theorem 1, of Levin (1984) to find a bounded distance function d on X such that (i) d metrizes X; and (ii) there is an $\mathcal{U}^2 \subseteq \mathrm{Lip}(X, d)$ such that

$$x \succsim y \quad \text{iff} \quad \mathcal{U}^2(x) \geq \mathcal{U}^2(y)$$

for every x and y in X. For each $u \in \mathcal{U}^2$, we set $L(u) := \max\{\|u\|_\infty, \mathrm{Lip}(u)\}$, and put $\mathcal{U}^3 := \{u/L(u) : u \in \mathcal{U}_2\}$. Obviously, $\mathcal{U}^3 \subseteq \mathrm{Lip}_1(X, d)$, and

$$x \succsim y \quad \text{iff} \quad \mathcal{U}^3(x) \geq \mathcal{U}^3(y)$$

[10] We owe the following argument to Kazuhiro Hara.

for every x and y in X. Now, $\mathcal{U}^3$ is equicontinuous, because any set of real Lipschitz maps on a metric space whose Lipschitz constants are bounded from above is equicontinuous. Moreover, $\mathcal{U}^3$ is bounded, because $\|u-v\|_\infty \leq 2$ for any u and v in $\mathcal{U}^3$. Therefore, by the Arzelà-Ascoli Theorem, $\mathcal{U}^3$ is relatively compact, that is, $\mathrm{cl}(\mathcal{U}^3)$ is compact, in $\mathbf{C}(X)$. But it is easily checked that

$$x \succsim y \quad \text{iff} \quad u(x) \geq u(y) \text{ for each } u \in \mathrm{cl}(\mathcal{U}^3)$$

for every x and y in X. Moreover, as $\mathrm{Lip}_1(X,d)$ is closed in $\mathbf{C}(X)$, we have $\mathrm{cl}(\mathcal{U}^3) \subseteq \mathrm{Lip}_1(X,d)$. Therefore, setting $\mathcal{U} := \mathrm{cl}(\mathcal{U}^3)$ completes our proof. ■

Proof of Proposition 4

The "if" part of the claim is straightforward. To establish its "only if" part, let $\mathbf{R}$ be a continuous, complete, and quasitransitive binary relation on X. Then, the binary relation $\mathbf{R}^> \cup \triangle_X$ is a partial order on X which is open in $X \times X$. So, by Theorem 2 of Sondermann (1980), we can find an upper semicontinuous function $u : X \to \mathbb{R}$ such that $u(x) > u(y)$ holds for every $x, y \in X$ with $x \; \mathbf{R}^> \; y$. It is without loss of generality to assume that $\|u\|_\infty < \frac{1}{2}$ here. (Otherwise, we would instead work with the map $\frac{1}{2\pi} \arctan u + \frac{1}{4}$.) Now, for any $z \in X$, define $v_z : X \to \mathbb{R}$ by

$$v_z(x) := \begin{cases} -1, & \text{if } z = x \text{ or } z \; \mathbf{R}^> \; x, \\ 0, & \text{otherwise,} \end{cases}$$

and put $\mathcal{V} := \{v_z + u : z \in X\} \cup \{u\}$. Then,

$$x \; \mathbf{R}^> \; y \quad \text{implies} \quad v(x) > v(y) \text{ for every } v \in \mathcal{V}. \tag{22}$$

(Indeed, if $x \; \mathbf{R}^> \; y$, then for an arbitrarily fixed $z \in X$ with $v_z(x) = -1$, we have either $z = x$ or $z \; \mathbf{R}^> \; x$, so we surely have $z \; \mathbf{R}^> \; y$, and hence $v_z(y) = -1$ so that $v_z(x) + u(x) > v_z(y) + u(y)$.) Furthermore,

$$x \; \mathbf{R}^= \; y \neq x \quad \text{implies} \quad v(x) > v(y) \text{ for some } v \in \mathcal{V}. \tag{23}$$

(Indeed, if $x \; \mathbf{R}^= \; y \neq x$, then $(v_y(x)+u(x)) - (v_y(y)+u(y)) = u(x) - u(y) + 1 > 0$ (because $\|u\|_\infty < \frac{1}{2}$), and hence, setting $v := v_y + u$ yields $v(x) > v(y)$.) To complete our proof, notice that for any x and y in X and any $v \in \mathcal{V}$, $v(x) - v(y)$ belongs to the finite set $\{u(x)-u(y)-1, u(x)-u(y), u(x)-u(y)+1\}$. It follows that $\max\{v(x) - v(y) : v \in \mathcal{V}\}$ is well-defined for any $x, y \in X$. As it is plain that (22) and (23) implies (9), we are done. ■

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